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# The Diurnal Variation of Terrestrial Magnetism

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# XV. *The Diurnal Variation of Terrestrial Magnetism.*

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## I. *Introduction.*

IN the year 1839 GAUSS published his celebrated Memoir on Terrestrial Magnetism, in which the potential on the Earth's surface was calculated to 26 terms of a series of surface harmonics. It was proved in this Memoir that, if the horizontal components of magnetic force were known all over the Earth, the surface potential could be derived without the help of the vertical forces, and it is well known now how these latter can be used to separate the terms of the potential which depend on internal from those which depend on external sources. Nevertheless GAUSS made use of the vertical forces in his calculations of the surface potential in order to ensure a greater degree of accuracy. He assumed for this purpose that magnetic matter was distributed through the interior of the Earth, and mentions the fair agreement between calculated and observed facts as a justification of his assumption. In the latter part of the Memoir it was suggested that the same method should be employed in the investigation of the regular and secular variations.

The use of harmonic analysis to separate internal from external causes has never been put to a practical test, but it seems to me to be especially well adapted to enquiries on the causes of the periodic oscillations of the magnetic needle.

If the magnetic effects can be fairly represented by a single term in the series of harmonics as far as the horizontal forces are concerned, there should be no doubt as to the location of the disturbing cause, for the vertical force should be in the opposite direction if the origin is outside from what it should be if the origin is inside the Earth. As the expression for the potential contains in one case the distance from the Earth's centre in the numerator, in the other case in the denominator, and as the vertical force depends on the differential coefficient with regard to distance from the Earth's centre, each single term in the series is of opposite sign according to the location of the cause; but what is true for each single term need not be true for the sum of the series. By a curious combination of terms the vertical forces might possibly be of the same sign, on whichever of the two hypotheses it is calculated. In any case, however, the differences between the two results will be of the same order of magnitude as the vertical force itself. If it is then a question simply of deciding whether the cause is outside or inside, without taking into account a possible combination of both causes, the result should not be doubtful, even if we have only an approximate knowledge of the vertical forces.

Two years ago I showed that the leading features of the horizontal components for diurnal variation could be approximately represented by the surface harmonic of the second degree and first type, and that the vertical variation agreed in direction and phase with the calculation on the assumption that the seat of the force is outside the Earth. The agreement seemed to me to be sufficiently good to justify the conclusion that the greater part of the variation is due to causes outside the Earth's surface. Nevertheless, it seemed advisable to enter more fully into the matter, as in the first approximate treatment of the subject a number of important questions had to be left untouched. I now publish the results of an investigation which has been carried as far as the observations at my disposal have allowed me to do. My original conclusions have been fully confirmed, and some further information has been obtained which I believe to be of importance. The results of the calculation point not only to an external source, but to an additional internal source, standing in fixed relationship to the external cause. This we might have expected. A varying potential due to external causes must be accompanied by currents induced in the Earth's body, which, in turn, must affect the magnetic needle. The phase of these currents and their magnitude lead us to form definite conclusions on the average conducting power of the Earth, and it will be seen that there is strong evidence that the average conductivity is very small near the surface, but must be greater further down. In this part of the investigation I had much assistance from my colleague, Professor LAMB.

I hope that the results obtained in this paper may induce the heads of magnetic observatories to consider the suggestions which I have made at the end of it, as their adoption would very materially assist further investigations.

I had, in the first place, to fix on a year for which we possess as complete magnetic records as possible. The phase of the variation of horizontal force changes sign in a latitude not far removed from that of Lisbon, and it seemed to me, therefore, essential that the excellent observations there made by Sen. JOS. CAPELLO should be made use of. The observations are published as far as 1872, and I had to take a year therefore anterior to this. It seemed also desirable to make use of the St. Petersburg observations, as it is the most northerly station for which we have records extending over a period of years, and as Mr. H. WILD's well-known skill gives special value to the observations made under his direction. The obseravtions (continued since 1878 in Paulowsk) were interrupted in 1871 and 1872, and we have to go back, therefore, as far as 1870 if we want to utilise the St. Petersburg and Lisbon observations simultaneously. As far as the horizontal components are concerned, we also possess good records of 1870 at Greenwich and Bombay. Four stations are sufficient to find to the necessary accuracy the potential on the surface of the Earth, but it would be of advantage if in future similar investigations a greater number of stations could be utilised.

The observations are published in very different form by the various observatories. The units of force at Bombay and St. Petersburg are the Gaussian unit millimeter-milligram-second. At Lisbon it is the foot-grain-second. At Greenwich the variations are given in terms of the whole vertical and horizontal force. At Bombay, moreover, the observations are given not for a certain hour, but for a time which varies with different instruments between 12 and 19 minutes past each hour. For this there was some reason originally, but at present it would be far better if Bombay would adopt the practice of other observatories. The daily variations had, in the first place, to be all reduced to C.G.S. units, and, further, instead of variations in declination and horizontal force, we had to find the components of the periodic force towards the geographical North and West. I am much indebted to Mr. WM. ELLIS, of the Greenwich Observatory, for the help he has given me in the reduction of the observations to a form which I could use in my calculations. A good many of the computations were done under his direct superintendence, and much time and trouble was saved me in consequence.

The daily variation of declination and horizontal force was expressed in the form

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \cos 4t,$$

where $t$ represents astronomical time. The summer months, April to September, were treated separately from the winter months, October to March. The unit of force, for convenience's sake, was taken as $10^{-6}$ C.G.S.

Tables I. and II. give the coefficients which were calculated according to a well-known method from the original observations.

Tables III. and IV. give the same coefficients reduced to forces directed to the geographical North and West, instead of to the magnetic North and West.

Before showing how, with the help of these coefficients, the surface potential can be calculated, I must deduce a few formulæ which will be used hereafter.

TABLE I.—Force to Magnetic West.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. $\times 10^{-6}$; $t$ being astronomical time.

| | Bombay. | | Lisbon. | | Greenwich. | | St. Petersburg. | |
|---|---|---|---|---|---|---|---|---|
| | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. |
| $a_1$ | + 64·3 | + 6·3 | + 125·5 | + 121·5 | + 168·0 | + 154·5 | + 109·0 | + 95·7 |
| $b_1$ | + 116·5 | + 33·2 | + 213·7 | + 132·6 | + 192·3 | + 113·5 | + 219·4 | + 96·7 |
| $a_2$ | + 134·3 | + 6·5 | + 126·1 | + 53·6 | + 129·1 | + 32·7 | + 104·2 | − 12·8 |
| $b_2$ | + 59·5 | + 23·2 | + 152·2 | + 111·1 | + 114·6 | + 84·0 | + 113·2 | + 69·6 |
| $a_3$ | + 103·5 | + 40·4 | + 75·1 | + 51·0 | + 65·8 | + 38·4 | + 37·9 | + 21·8 |
| $b_3$ | + 7·1 | + 7·6 | + 56·2 | + 51·0 | + 42·5 | + 30·6 | + 59·2 | + 28·9 |
| $a_4$ | + 15·5 | + 33·1 | + 13·1 | + 30·7 | + 9·3 | + 21·3 | + 2·8 | + 6·2 |
| $b_4$ | + 18·7 | − 9·6 | − 5·9 | + 20·3 | + 2·6 | + 17·6 | + 10·9 | + 13·7 |

TABLE II.—Force to Magnetic North.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. $\times 10^{-6}$; $t$ being astronomical time.

| | Bombay. | | Lisbon. | | Greenwich. | | St. Petersburg. | |
|---|---|---|---|---|---|---|---|---|
| | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. |
| $a_1$ | + 386·9 | + 332·5 | − 124·2 | − 78·6 | − 213·7 | − 117·4 | − 257·9 | − 111·8 |
| $b_1$ | + 5·7 | + 38·2 | + 71·5 | − 15·0 | + 154·0 | + 22·8 | + 150·2 | − 5·8 |
| $a_2$ | + 152·7 | + 125·0 | − 28·3 | − 54·8 | − 112·6 | − 87·5 | − 149·1 | − 69·0 |
| $b_2$ | − 11·8 | + 7·3 | + 43·0 | − 0·9 | + 78·9 | + 24·6 | + 45·7 | − 12·8 |
| $a_3$ | + 41·1 | + 39·6 | + 23·1 | − 2·3 | − 2·3 | − 16·4 | − 15·8 | − 18·4 |
| $b_3$ | − 41·8 | − 31·4 | + 19·0 | + 19·0 | + 27·1 | + 27·1 | + 39·7 | + 22·9 |
| $a_4$ | − 4·3 | + 15·3 | + 12·2 | + 6·9 | + 12·1 | + 2·0 | − 9·0 | − 3·8 |
| $b_4$ | − 13·7 | − 21·1 | − 12·3 | + 18·8 | + 8·4 | + 16·6 | + 3·3 | + 6·4 |

TABLE III.—Force to Geographical West.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. $\times 10^{-6}$; $t$ being astronomical time.

| | Bombay. | | Lisbon. | | Greenwich. | | St. Petersburg. | |
|---|---|---|---|---|---|---|---|---|
| | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. |
| $a_1$ | + 57·5 | + 0·4 | + 74·3 | + 86·5 | + 85·4 | + 105·4 | + 99·6 | + 91·6 |
| $b_1$ | + 116·4 | + 32·5 | + 225·3 | + 119·1 | + 233·3 | + 114·6 | + 224·6 | + 96·4 |
| $a_2$ | + 131·6 | + 4·3 | + 108·3 | + 31·1 | + 83·1 | + 1·0 | + 98·7 | − 15·3 |
| $b_2$ | + 59·7 | + 23·1 | + 157·7 | + 103·8 | + 134·6 | + 87·4 | + 114·8 | + 69·1 |
| $a_3$ | + 102·8 | + 39·7 | + 78·5 | + 47·0 | + 61·1 | + 30·5 | + 37·3 | + 21·2 |
| $b_3$ | + 7·8 | + 8·1 | + 59·3 | + 54·4 | + 49·2 | + 38·0 | + 60·6 | + 29·7 |
| $a_4$ | + 15·6 | + 32·8 | + 16·6 | + 31·2 | + 12·9 | + 20·7 | + 2·5 | + 6·1 |
| $b_4$ | − 18·5 | − 9·2 | − 9·8 | + 25·6 | + 5·3 | + 22·2 | + 11·0 | + 13·9 |

TABLE IV.—Force to Geographical North.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. $\times\, 10^{-6}$; $t$ being astronomical time.

| | Bombay. | | Lisbon. | | Greenwich. | | St. Petersburg. | |
|---|---|---|---|---|---|---|---|---|
| | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. |
| $a_1$ | + 388·0 | + 332·4 | − 160·2 | − 116·1 | − 258·1 | − 162·9 | − 261·6 | − 115·2 |
| $b_1$ | + 7·8 | + 38·8 | − 7·6 | − 60·4 | + 79·5 | − 17·2 | + 142·2 | − 9·3 |
| $a_2$ | + 155·1 | + 125·1 | − 70·5 | − 70·1 | − 149·8 | − 93·4 | − 152·8 | − 68·5 |
| $b_2$ | − 10·8 | + 7·7 | − 12·8 | − 39·6 | + 35·2 | − 5·5 | + 41·6 | − 15·3 |
| $a_3$ | + 42·9 | + 40·3 | − 4·5 | − 20·0 | − 24·6 | − 28·5 | − 17·2 | − 19·2 |
| $b_3$ | − 41·7 | − 31·3 | − 1·8 | 0·0 | + 11·0 | + 15·1 | + 37·6 | + 21·9 |
| $a_4$ | − 4·0 | + 15·9 | + 6·8 | − 4·2 | + 8·2 | − 5·3 | − 9·1 | − 4·0 |
| $b_4$ | − 14·0 | − 21·3 | − 9·5 | + 10·5 | + 7·0 | + 9·6 | + 2·9 | + 5·9 |

## II. *Some Formulæ useful in the Analysis by Spherical Harmonics.*

In the expansion of mathematical expressions into spherical harmonics, it will often occur that we have to express powers of the cosine of an angle, or a cosine of some multiple of an angle, in terms of the differential coefficients of zonal harmonics with respect to the cosine of the argument. The necessary equations for the powers of cosines can be easily obtained by differentiation of the well-known formulæ, giving the powers of a cosine in terms of zonal harmonics. But I think it will be useful here to give the general equations which I have had to use in expressing $\cos mu$ in terms of $d^p P_i / d\mu^p$, where $p$ is any given number, $\mu = \cos u$ and $P_i$ the zonal harmonic of degree $i$.

We may start from the expression

$$\cos mu = A_m P_m + A_{m-2} P_{m-2} + \ldots A_i P_i + \ldots \qquad (1),$$

where

$$A_i = -(2i+1)\frac{\{m-(i-2)\}\{m-(i-4)\}\ldots(m-2)\,m^2\,(m+2)\ldots\{m+(i-2)\}}{\{m-(i+1)\}\{m-(i-1)\}\ldots\,(m-1)(m+1)\,\ldots\{m+(i+1)\}},$$

if $m$ and $i$ be even, and

$$A_i = -(2i+1)\frac{\{m-(i-2)\}\{m-(i-4)\}\ldots(m-1)(m+1)\ldots\{m+(i-2)\}}{\{m-(i+1)\}\{m-(i-1)\}\ldots(m-2)(m+2)\ldots\{m+(i+1)\}},$$

if $m$ and $i$ be odd.

Both expressions are included in the general one

$$A_i = -(2i+1)\, m \frac{\{m-(i-2)\}\{m-(i-4)\}\ldots\{m+(i-2)\}}{\{m-(i+1)\}\{m-(i-1)\}\ldots\{m+(i+1)\}},$$

where successive factors of both numerator and denominator increase by 2.*

If in formula (1) we substitute for any term $P_i$

$$\frac{dP_{i+1}}{d\mu} - \frac{dP_{i-1}}{d\mu} = (2i+1)\, P_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

we obtain an expression for $\cos m\theta$ in terms of the first differential coefficients of the zonal harmonics; and, if in the formula so obtained we successively apply the transformation

$$\frac{d^pP_{i+1}}{d\mu^p} - \frac{d^pP_{i-1}}{d\mu^p} = (2i+1)\frac{d^{p-1}P_i}{d\mu^{p-1}},$$

we finally obtain the required expression in terms of any required differential coefficient.

I have obtained in this way the equation

$$(-1)^{p+1}\frac{\cos mu}{1.3.5\ldots(2p+1)} = A_{m+p}\, d^pP_{m+p} + A_{m+p-2}\, d^pP_{m+p-2} + \ldots A_i\, d^pP_i + \ldots \quad (3),$$

where $d^pP_i$ is written shortly for $d^pP_i/d\mu^p$, and

$$A_i = (2i+1)\, m \frac{\{m-(i-p-2)\}\{m-(i-p-4)\}\ldots\{m+(i-p-2)\}}{\{m-(i+p+1)\}\{m-(i+p-1)\}\ldots\{m+(i+p+1)\}} \quad . \quad (4),$$

except for the last term of the series, which will be given presently (5). If $p = 0$, this expression agrees with the one previously given, and, as I shall proceed to show, if it is true for any value $p$, it will also be true for the value $p+1$. Assume, then, the equation (3) to hold.

The relation

$$d^{p+1}P_{(i+1)} - d^{p+1}P_{(i-1)} = (2i+1)\, d^pP_i$$

shows that the factor multiplying $d^{p+1}P_{i+1}$ in the expression for $\cos m\theta$ will depend only on $A_i$ and on $A_{(i+2)}$ in (3). If $B_{(i+1)}$ is this factor, we have

$$B_{(i+1)} = \frac{A_i}{2i+1} - \frac{A_{i+2}}{2i+5},$$

* In the very useful book on 'Spherical Harmonics,' by FERRERS, the factor $m$ does not occur in the general expression for $A_i$ (page 33); but, from the deduction of the formula, it is clear that the factor $m$ must be taken twice when it occurs in the numerator, and not at all when it occurs in the denominator. In using the equation I was at first led into error by the ambiguity, and hence I believe the expression given above to be clearer.

or, by substituting $A_i$ from (4) and $A_{i-2}$ from the corresponding equation,

$$B_{i+1} = m \frac{\{m-(i-p-2)\}\{m-(i-p-4)\}\ldots\{m+(i-p-2)\}}{\{m-(i+p+1)\}\{m-(i-p-1)\}\ldots\{m+(i+p+1)\}} \left[1 - \frac{\{m-(i-p)\}\{m+(i-p)\}}{\{m-(i+p+3)\}\{m+(i+p+3)\}}\right].$$

The square bracket reduces to

$$\frac{(i-p)^2-(i+p+3)^2}{\{m-(i+p+3)\}\{m+(i+p+3)\}} = -\frac{(2i+3)(2p+3)}{\{m-(i+p+3)\}\{m+(i+p+3)\}},$$

so that

$$-(-1)^{p+1}\frac{\cos m\theta}{1.3.5\ldots(2p+3)} = A_{m+p+1}\,d^{p+1}P_{m+p+1} + \ldots A_{i+1}\,d^{p+1}P_{i+1}\ldots,$$

where

$$A_{i+1} = m(2i+3)\frac{\{m-(i-p-2)\}\{m-(i-p-4)\}\ldots\{m+(i-p-2)\}}{\{m-(i+p+3)\}\{m-(i+p+1)\}\ldots\{m+(i+p+3)\}},$$

which agrees with (4) if $p+1$ is written for $p$, and $i+1$ for $i$.

The end terms require special consideration.

As only the even or only the odd zonal harmonics enter into any one series, the difference between $m+p$ and $i$ in the equation (3) must be even; hence, the numerators in the fractional expression of (4) consist always of even numbers, whatever values $m$, $i$, or $p$ may have.

If $(m+p)$ be odd, the zonal harmonics will all be odd, and the last term of the series will depend on $d^pP_{p+1}$ if $p$ be even, and $d^pP_p$ if $p$ be odd, for the differential coefficients of the zonal harmonics of lower degree will vanish. The expression (4) in neither case gives the correct factor, and we must substitute for it in both cases

$$(2i+1)\frac{1}{\{m-(i+p+1)\}\{m-(i+p-1)\}\ldots\{m+(i+p+1)\}} \quad . \quad . \quad (5).$$

If in (4) $i$ is put equal to $(p+2)$, the first and last term will be equal to $m$; in that case the $m$ is only taken once.

The expression (5) is easily proved, and shown to hold also if $(m+p)$ be even.

The first term of the series is included in the general expression (4).

I have deduced with the help of the equation (3) the following relations, which will be used in this paper:—

$$\cos u = \tfrac{1}{3}\frac{dP_2}{d\mu} \quad \ldots \quad (A),$$

$$\cos 2u = \tfrac{4}{15}\frac{dP_3}{d\mu} - \tfrac{9}{15}\frac{dP_1}{d\mu} \quad \ldots \quad (B),$$

$$\cos 3u = \tfrac{8}{35}\frac{dP_4}{d\mu} - \tfrac{3}{7}\frac{dP_2}{d\mu} \quad \ldots \quad (C),$$

$$\cos 4u = \tfrac{64}{315}\frac{dP_5}{d\mu} - \tfrac{16}{45}\frac{dP_3}{d\mu} + \tfrac{3}{35}\frac{dP_1}{d\mu} \quad \ldots \quad (D),$$

$$\cos 5u = \tfrac{128}{693}\frac{dP_6}{d\mu} - \tfrac{24}{77}\frac{dP_4}{d\mu} + \tfrac{5}{63}\frac{dP_2}{d\mu} \quad \ldots \quad (E),$$

$$\cos 6u = \tfrac{512}{3003}\frac{dP_7}{d\mu} - \tfrac{128}{455}\frac{dP_5}{d\mu} + \tfrac{4}{55}\frac{dP_3}{d\mu} + \tfrac{1}{105}\frac{dP_1}{d\mu} \quad \ldots \quad (F);$$

$$\cos u = \tfrac{1}{105}\frac{d^3P_4}{d\mu^3} \quad \ldots \quad (G),$$

$$\cos 2u = \tfrac{4}{945}\frac{d^3P_5}{d\mu^3} - \tfrac{7}{135}\frac{d^3P_3}{d\mu^3} \quad \ldots \quad (H),$$

$$\cos 3u = \tfrac{8}{3465}\frac{d^3P_6}{d\mu^3} - \tfrac{1}{55}\frac{d^3P_4}{d\mu^3} \quad \ldots \quad (I),$$

$$\cos 4u = \tfrac{64}{45045}\frac{d^3P_7}{d\mu^3} - \tfrac{16}{1755}\frac{d^3P_5}{d\mu^3} + \tfrac{7}{297}\frac{d^3P_3}{d\mu^3} \quad \ldots \quad (K),$$

$$\cos 5u = \tfrac{128}{135135}\frac{d^3P_8}{d\mu^3} - \tfrac{8}{1485}\frac{d^3P_6}{d\mu^3} + \tfrac{5}{429}\frac{d^3P_4}{d\mu^3} \quad \ldots \quad (L),$$

$$\cos 6u = \tfrac{512}{765765}\frac{d^3P_9}{d\mu^3} - \tfrac{128}{36465}\frac{d^3P_7}{d\mu^3} + \tfrac{4}{585}\frac{d^3P_5}{d\mu^3} - \tfrac{7}{1287}\frac{d^3P_3}{d\mu^3} \quad \ldots \quad (M).$$

There is another formula useful in similar investigations, which may find a place here, although I have not used it in the final reductions. It is the expression of a zonal harmonic in terms of the $n^{\text{th}}$ differential coefficient of other zonal harmonics.

$$\begin{aligned} P_i = & \frac{(2i+2n+1)}{(2i+2n+1)(2i+2n-1)\ldots(2i+1)}\, d^nP_{i+n} \\ & - \frac{n}{1}\frac{(2i+2n-3)}{(2i+2n-1)(2i+2n-3)\ldots(2i-1)}\, d^nP_{i+n-2} \\ & + \frac{n.n-1}{1.2}\cdot\frac{(2i+2n-7)}{(2i+2n-3)\ldots(2i-3)}\, d^nP_{i+n-4}\ldots \\ & \pm \frac{2i-(2n-1)}{(2i+1)(2i-1)\ldots(2i-(2n-1))}\, d^nP_{i-n}, \end{aligned}$$

the general $p^{\text{th}}$ term being

$$(-1)^{p-1}\frac{n.n-1\ldots n-(p-2)}{1.2\ldots p-1}\;\frac{2i+2n+5-4p}{(2i+2n+3-2p)(2i+2n+1-2p)\ldots(2i+3-2p)}\, d^nP_{i+n-2p+4}.$$

The proof is conducted exactly in the same way as for equation (3). If $i$ is equal to or greater than $2n$, the series has its full number of terms, viz., $n+1$, otherwise the series breaks off owing to the differential coefficients vanishing.

### III. *Expansion of Potential in terms of a Series of Surface Harmonics.*

We know from observation that, excepting the Arctic regions, the daily variation of the West force is nearly the same along the same circle of latitude. It is this fact which renders the present investigation possible, as comparatively few places of observation will be necessary to give us a very fair idea of the nature of the oscillation over a considerable area of the Earth's surface. If at any place we find that the daily oscillation of any one element can be expressed in the form

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \ldots,$$

when $t$ is reckoned by local time, we may get the variation at any point of the same latitude circle by writing $t + \lambda$ for $t$, where $\lambda$ is the longitude towards the East from some standard meridian and $t$ now is the time of the standard meridian. At the time $t = 0$ we have then the variation of the force to geographical West in different longitudes expressed by

$$\mathrm{Y} = a_1 \cos \lambda + b_1 \sin \lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda.$$

The coefficients will be functions of the latitude, and by expressing these functions in a series of proper form we may at once obtain an expression for the potential.

If X is the force to geographical North, Y the force to geographical West, and Z the vertical force, reckoned positive upwards, we have, putting $u$ for the colatitude, and $\lambda$ for the longitude towards the East,

$$\mathrm{X} = \frac{d\mathrm{V}}{a\,du}, \qquad \mathrm{Y} \sin u = \frac{d\mathrm{V}}{a\,d\lambda}, \qquad \mathrm{Z} = -\frac{d\mathrm{V}}{dr},$$

$a$ being the Earth's radius.

If we can expand $\mathrm{Y} \sin u$ in terms of surface harmonics our task is accomplished, and for this purpose we need only express $a_1 \sin u$, $b_1 \sin u$, &c., in a series of tesseral harmonics.

The expansion of a function of an angle in terms of the trigonometrical functions of its multiples is so easily carried out by the method of least squares, if the function is given for a regular series of submultiples of $2\pi$, that it seemed to me to be the easiest method of proceeding to obtain first by interpolation from the observed coefficients of Y its values for equidistant circles of latitude.

To obtain a curve for each of the values $a_1$, $b_1$, $a_2$, $b_2$, as depending on the latitude, we have the following data. The values are directly observed for four points in the Northern hemisphere. Taking the potential (as it is observed to be) symmetrical on the Northern and Southern hemispheres, and fixing the period to which we apply the calculation to be the one for which the mean value of the observed summer variations in the Northern hemisphere holds good, we must put, for the West force at the same period, in the Southern half of the globe, the observed *winter* values of the corresponding Northern latitudes, reversing, however, the sign to make our expression agree with observation. We have then eight values for the different terms of Y through which we might at once proceed to draw a curve; but, making use of the observed variations of Northern force, we can also calculate the direction of the tangent of the required curve for the same eight points. This has been done as follows. Let Y be expressed as

$$\Sigma(a_n \cos n\lambda + b_n \sin n\lambda),$$

and X as

$$\Sigma(\alpha_n \cos n\lambda + \beta_n \sin n\lambda).$$

The existence of a potential implies the relation

$$\frac{d\text{Y} \sin u}{du} = \frac{d\text{X}}{d\lambda}$$

or

$$\frac{d\text{Y}}{du} = \frac{d\text{X}}{d\lambda} \operatorname{cosec} u - \text{Y} \cot u,$$

from which, by substitution,

$$\Sigma\left(\frac{da_n}{du} \cos n\lambda + \frac{db_n}{du} \sin n\lambda\right)$$
$$= \Sigma[(n\beta_n \operatorname{cosec} u - a_n \cot u) \cos n\lambda - (n\alpha_n \operatorname{cosec} u + b_n \cot u) \sin n\lambda];$$

and, therefore,

$$\frac{da_n}{du} = n\beta_n \operatorname{cosec} u - a_n \cot u,$$

$$\frac{db_n}{du} = -\, n\alpha_n \operatorname{cosec} u + b_n \cot u.$$

These equations give the rate of change per radian; to get the rate of change per degree of colatitude we have to multiply the differential coefficients with the circular measure of one degree. The following Table gives the rate of change of the West force per degree of colatitude, calculated as explained from the North force.

TABLE V.

The unit of force in this Table is $10^{-6}$ C.G.S., and the unit of latitude is 1 degree.

| | Bombay. | | Lisbon. | | Greenwich. | | St. Petersburg. | |
|---|---|---|---|---|---|---|---|---|
| | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. | Summer. | Winter. |
| $da_1/du$ . . . | − 0·201 | + 0·714 | − 1·210 | − 2·561 | + 0·356 | − 2·793 | + 1·951 | − 3·086 |
| $db_1/du$ . . . | − 7·853 | − 6·326 | + 0·432 | + 0·931 | + 2·118 | + 2·053 | + 2·342 | + 1·107 |
| $da_2/du$ . . . | − 1·184 | + 0·258 | − 2·087 | − 2·207 | + 0·150 | − 0·330 | − 0·079 | − 0·605 |
| $db_2/du$ . . . | − 6·079 | − 4·754 | + 0·948 | + 1·684 | + 5·445 | + 3·318 | + 7·185 | + 2·689 |
| $da_3/du$ . . . | − 2·921 | − 1·968 | − 1·218 | − 0·658 | − 0·416 | + 0·600 | + 2·805 | + 1·650 |
| $db_3/du$ . . . | − 2·420 | − 2·277 | − 0·527 | + 0·580 | + 0·988 | + 1·564 | − 0·030 | + 1·112 |
| $da_4/du$ . . . | − 1·125 | − 1·768 | − 1·084 | + 0·503 | + 0·501 | + 0·622 | + 0·329 | + 0·640 |
| $db_4/du$ . . . | + 0·407 | − 1·117 | − 0·471 | + 0·018 | − 1·036 | + 0·109 | + 0·936 | + 0·137 |

Curves were now carefully drawn for each of the eight coefficients, making them fit in as well as possible with the ordinates and the direction of their tangents, as given in Tables III. and V.

The values of Y were then read off for each 7°·5 of colatitude, and a fresh table was formed (Table VI.).

From this point onwards we have to carry on the calculations separately for each type of the variation.

TABLE VI.

Coefficients in the series $Y = a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t +$ &c. for different degrees of colatitude, the unit of force being C.G.S. $\times 10^{-6}$.

| Colatitude. | $a_1$. | $b_1$. | $a_2$. | $b_2$. | $a_3$. | $b_3$. | $a_4$. | $b_4$. |
|---|---|---|---|---|---|---|---|---|
| 0° | + 10 | 0 | 0 | 0 | 0 | (+ 48) | ( 0) | ( 0) |
| 7·5 | + 32 | + 97 | + 42 | + 9 | (+ 12) | (+ 51) | (+ 1) | (+ 3) |
| 15·0 | + 52 | + 158 | + 76 | + 22 | (+ 24) | (+ 53) | (+ 1) | (+ 6) |
| 22·5 | + 71 | + 196 | + 91 | + 48 | + 39 | + 54 | + 1 | + 7 |
| 30·0 | + 85 | + 215 | + 90 | + 101 | + 56 | + 54 | + 2 | + 11 |
| 37·5 | + 96 | + 231 | + 89 | + 143 | + 73 | + 55 | + 11 | + 18 |
| 45·0 | + 92 | + 243 | + 96 | + 160 | + 99 | + 61 | + 16 | − 3 |
| 52·5 | + 83 | + 247 | + 90 | + 168 | + 98 | + 58 | + 22 | − 21 |
| 60·0 | + 74 | + 205 | + 83 | + 165 | + 91 | + 54 | + 20 | − 24 |
| 67·5 | + 65 | + 144 | + 131 | + 83 | + 89 | + 28 | + 11 | − 20 |
| 75·0 | + 63 | + 87 | + 127 | + 37 | + 69 | + 10 | + 2 | − 15 |
| 82·5 | + 58 | + 58 | + 117 | + 15 | + 46 | − 3 | − 6 | − 11 |
| 90·0 | + 37 | + 42 | + 82 | + 7 | + 25 | − 2 | − 14 | + 6 |
| 97·5 | + 10 | + 21 | + 15 | + 1 | + 3 | + 1 | − 20 | + 18 |
| 105·0 | − 3 | − 7 | − 4 | − 17 | − 16 | − 6 | − 27 | + 13 |
| 112·5 | + 3 | − 56 | − 3 | − 48 | − 36 | − 23 | − 40 | 0 |
| 120·0 | 0 | − 105 | − 4 | − 102 | − 47 | − 45 | − 43 | − 23 |
| 127·5 | − 50 | − 120 | − 15 | − 118 | − 53 | − 50 | − 35 | − 26 |
| 135·0 | − 78 | − 113 | − 16 | − 106 | − 54 | − 46 | − 31 | − 25 |
| 142·5 | − 97 | − 100 | + 10 | − 85 | − 47 | − 37 | − 24 | − 24 |
| 150·0 | − 121 | − 85 | + 6 | − 65 | − 38 | − 30 | − 12 | − 14 |
| 157·5 | − 140 | − 75 | + 2 | − 46 | − 22 | − 21 | − 5 | − 13 |
| 165·0 | − 154 | − 61 | + 1 | − 30 | (− 9) | (− 14) | (− 3) | (− 9) |
| 172·5 | − 162 | − 36 | 0 | − 14 | (− 2) | (− 6) | (− 2) | (− 4) |
| 180·0 | − 165 | 0 | 0 | 0 | ( 0) | ( 0) | ( 0) | ( 0) |

The quantities $a_1$, $b_1$ were expressed in the usual way in terms of the multiples of the cosines of the colatitude, and two equations obtained which can be shown to represent with sufficient accuracy the force of the first type towards the geographical West.

$$a_1 = -5 + 106{\cdot}7 \cos u - 50{\cdot}6 \cos 2u - 1{\cdot}7 \cos 3u - 14{\cdot}6 \cos 4u - 8{\cdot}9 \cos 5u - 2{\cdot}0 \cos 6u,$$

$$b_1 = 49{\cdot}4 + 154{\cdot}3 \cos u - 0{\cdot}4 \cos 2u - 76{\cdot}0 \cos 3u - 18{\cdot}1 \cos 4u - 31{\cdot}3 \cos 5u - 9{\cdot}2 \cos 6u.$$

If, now, $\cos u$, $\cos 2u$, $\cos 3u$, &c., be expressed in terms of $dP_1/d\mu$, $dP_2/d\mu_1$, $dP_3/d\mu$,

&c., we have an equation for that part of Y which depends on $\cos\lambda$ and $\sin\lambda$. After multiplication with $\sin u$, one side of the equation contains the colatitude in form of tesseral harmonics only, and hence we obtain at once the required expansion of $V^{(1)}$. The equations are

$$Y^{(1)} = \cos\lambda\left[24{\cdot}09\frac{dP_1}{d\mu} + 35{\cdot}59\frac{dP_2}{d\mu} - 8{\cdot}45\frac{dP_3}{d\mu} + 2{\cdot}39\frac{dP_4}{d\mu} - 2{\cdot}41\frac{dP_5}{d\mu} - 1{\cdot}64\frac{dP_6}{d\mu}\right]$$

$$+ \sin\lambda\left[48{\cdot}00\frac{dP_1}{d\mu} + 81{\cdot}52\frac{dP_2}{d\mu} + 5{\cdot}66\frac{dP_3}{d\mu} - 7{\cdot}62\frac{dP_4}{d\mu} - 1{\cdot}09\frac{dP_5}{d\mu}\right.$$

$$\left. - 5{\cdot}78\frac{dP_6}{d\mu} - 1{\cdot}57\frac{dP_7}{d\mu} - 0{\cdot}34\frac{dP_7}{d\mu}\right].$$

With the help of

$$\frac{dV}{d\lambda} = Ya\sin u,$$

we have, finally,

$$-V^{(1)}/a = \cos\lambda[48{\cdot}00T_1{}^1 + 81{\cdot}52T_2{}^1 + 5{\cdot}66T_3{}^1 - 7{\cdot}62T_4{}^1$$

$$- 1{\cdot}09T_5{}^1 - 5{\cdot}78T_6{}^1 - 1{\cdot}57T_7{}^1]$$

$$+ \sin\lambda[-24{\cdot}09T_1{}^1 + 35{\cdot}59T_2{}^1 + 8{\cdot}45T_3{}^1 - 2{\cdot}39T_4{}^1$$

$$+ 2{\cdot}41T_5{}^1 + 1{\cdot}64T_6{}^1 + 0{\cdot}34T_7{}^1] \quad . \quad . \quad [A].$$

To obtain, similarly, an expression for $V^{(2)}$, the series of coefficients $a_2$, $b_2$ were expressed in terms of $\sin u$, $\sin 2u$, &c., giving the equations

$$a_2 = 67{\cdot}0\sin u + 61{\cdot}3\sin 2u + 7{\cdot}4\sin 3u - 11{\cdot}6\sin 4u + 15{\cdot}9\sin 5u + 18{\cdot}6\sin 6u,$$

$$b_2 = 21{\cdot}5\sin u + 113{\cdot}2\sin 2u + 7{\cdot}7\sin 3u - 12{\cdot}6\sin 4u - 10{\cdot}8\sin 5u - 28{\cdot}2\sin 6u.$$

From the known expansion of $\cos pu$, in terms of the first differential coefficient of the zonal harmonics, we can obtain by differentiation an equation which will at once give us $V^{(2)}$ in the required form.

Thus, for instance, we have by Equation D, page 474,

$$\cos 4u = \tfrac{64}{315}\frac{dP_5}{d\mu} - \tfrac{16}{45}\frac{dP_3}{d\mu} + \tfrac{3}{35}\frac{dP_1}{d\mu},$$

and differentiating with respect to $\mu$,

$$4\sin 4u / \sin u = \tfrac{64}{315}\frac{d^2P_5}{d\mu^2} - \tfrac{16}{45}\frac{d^2P_3}{d\mu^2} + \tfrac{3}{35}\frac{d^2P_1}{d\mu^2}.$$

By substituting this and other similar expressions, we find

$$\mathrm{Y}^{(2)}/\sin u = \cos 2\lambda\left[64\cdot59 + 9\cdot42\frac{d^2\mathrm{P}_3}{d\mu^2} - 0\cdot42\frac{d^2\mathrm{P}_4}{d\mu^2} - 1\cdot47\frac{d^2\mathrm{P}_5}{d\mu^2} + 0\cdot59\frac{d^2\mathrm{P}_6}{d\mu^2} + 0\cdot51\frac{d^2\mathrm{P}_7}{d\mu^2}\right]$$
$$+ \sin 2\lambda\left[17\cdot69 + 15\cdot87\frac{d^2\mathrm{P}_3}{d\mu^2} + 1\cdot27\frac{d^2\mathrm{P}_4}{d\mu^2} + 0\cdot68\frac{d^2\mathrm{P}_5}{d\mu^2} - 0\cdot40\frac{d^2\mathrm{P}_6}{d\mu^2} - 0\cdot80\frac{d^2\mathrm{P}_7}{d\mu^2}\right],$$

and from this, by multiplication with $\sin^2 u$ and integration with respect to $\lambda$,

$$-\mathrm{V}^{(2)}/a = \cos 2\lambda\,[2\cdot95\,\mathrm{T}_2^2 + 7\cdot94\,\mathrm{T}_3^2 + 0\cdot63\,\mathrm{T}_4^2 + 0\cdot34\,\mathrm{T}_5^2 - 0\cdot20\,\mathrm{T}_6^2 - 0\cdot41\,\mathrm{T}_7^2]$$
$$-\sin 2\lambda\,[10\cdot76\,\mathrm{T}_2^2 + 4\cdot72\,\mathrm{T}_3^2 - 0\cdot21\,\mathrm{T}_4^2 - 0\cdot73\,\mathrm{T}_5^2 + 0\cdot29\,\mathrm{T}_6^2 + 0\cdot26\,\mathrm{T}_7^2]\,.\ [\mathrm{B}].$$

To find $\mathrm{V}^{(3)}$ from the coefficients $a_3$, $b_3$, $a_4$, $b_4$, the values of these coefficients were all divided by $\sin^2 u$, for a reason which will appear. It was then found, however, that to represent the numbers so obtained satisfactorily, a greater number of terms was necessary in the expansion than the observations seemed to justify. I was thus led to give up calculating as far as these types are concerned the terms on which the difference between summer and winter depends, so that, instead of using the coefficients from Table VI., the mean of corresponding values on both sides of the equator were taken, changing, of course, the sign of the coefficient given for the Southern hemisphere. In the next place, it was found that, owing to the smallness of $\sin u$ at high latitude, the three numbers corresponding to the colatitudes 0°, 7°·5, 15° were very large, while their weight is very small, as they have only been obtained by graphical extrapolation; I have, therefore, discarded these numbers altogether, putting them into brackets in Table VI. BESSEL has shown how the method of least squares may be applied to obtain a trigonometrical series for a succession of values some of which are missing.

The equations thus calculated are as follows :—

$$a_3/\sin^2 u = 74\cos u - 118\cos 3u - 101\cos 5u,$$
$$b_3/\sin^2 u = 23\cos u - 111\cos 3u - 127\cos 5u,$$
$$a_4/\sin^2 u = 39\sin 2u - 6\sin 4u - 9\sin 6u,$$
$$b_4/\sin^2 u = 31\sin 2u + 38\sin 4u + 4\sin 6u.$$

The agreement with the numbers from which the series are obtained is not as good as would be desirable, especially for $a_3$ and $b_3$, but the types higher than the second will probably depend much on local circumstances, and the result would not, in my opinion, repay the trouble of taking account of further terms in the above series; $\cos u$, $\cos 3u$, and $\cos 5u$ have already been given in terms of $d^3\mathrm{P}_i/d\mu^3$ (Equations G, I, L, page 474), and, by differentiation with respect to $u$, we can also get $\sin 2u$, sin $4u$, sin $6u$ in terms of $\sin u \, d^4 P_i / d\mu^4$. This gives, by substitution and integration with respect to λ,

$$- V^{(3)}/a = (- 0{\cdot}0401\, T_8{}^3 + 0{\cdot}1426\, T_6{}^3 + 0{\cdot}2523\, T_4{}^3) \cos 3\lambda$$
$$+ (0{\cdot}0319\, T_8{}^3 - 0{\cdot}0936\, T_6{}^3 - 0{\cdot}5577\, T_4{}^3) \sin 3\lambda$$
$$+ (0{\cdot}00033\, T_9{}^4 + 0{\cdot}00279 T_7{}^4 - 0{\cdot}00411\, T_5{}^4) \cos 4\lambda$$
$$+ (- 0{\cdot}00075\, T_9{}^4 + 0{\cdot}00078\, T_7{}^4 - 0{\cdot}01978\, T_5{}^4) \sin 4\lambda.$$

We have obtained thus finally, 38 coefficients in the expansion of V/$a$, which, for the sake of reference, I collect into Table VII.

In this Table $C_n{}^m$ is the coefficient of $T_n{}^m \cos m\lambda$.

$S_n{}^m$ „ „ $T_n{}^m \sin m\lambda$.

TABLE VII.

The unit of force is C.G.S. $10^{-6}$.

| | | | | | |
|---|---|---|---|---|---|
| $C_1{}^1$ | − 48·00 | $C_2{}^2$ | − 2·948 | $C_4{}^3$ | − 0·252 |
| $C_2{}^1$ | − 81·52 | $C_3{}^2$ | − 7·936 | $C_6{}^3$ | − 0·143 |
| $C_3{}^1$ | − 5·66 | $C_4{}^2$ | − 0·630 | $C_8{}^3$ | + 0·040 |
| $C_4{}^1$ | + 7·62 | $C_5{}^2$ | − 0·341 | | |
| $C_5{}^1$ | + 1·09 | $C_6{}^2$ | + 0·200 | $C_5{}^4$ | + 0·00411 |
| $C_6{}^1$ | + 5·78 | $C_7{}^2$ | + 0·407 | $C_7{}^4$ | − 0·00279 |
| $C_7{}^1$ | + 1·57 | | | $C_9{}^4$ | − 0·00033 |
| $S_1{}^1$ | + 24·09 | $S_2{}^2$ | + 10·764 | $S_4{}^3$ | + 0·578 |
| $S_2{}^1$ | + 35·59 | $S_3{}^2$ | + 4·715 | $S_6{}^3$ | + 0·094 |
| $S_3{}^1$ | − 8·45 | $S_4{}^2$ | − 0·214 | $S_8{}^3$ | − 0·032 |
| $S_4{}^1$ | + 2·39 | $S_5{}^2$ | − 0·731 | | |
| $S_5{}^1$ | − 2·41 | $S_6{}^2$ | + 0·293 | $S_5{}^4$ | + 0·01978 |
| $S_6{}^1$ | − 1·64 | $S_7{}^2$ | + 0·264 | $S_7{}^4$ | − 0·00078 |
| $S_7{}^1$ | − 0·34 | | | $S_9{}^4$ | + 0·00075 |

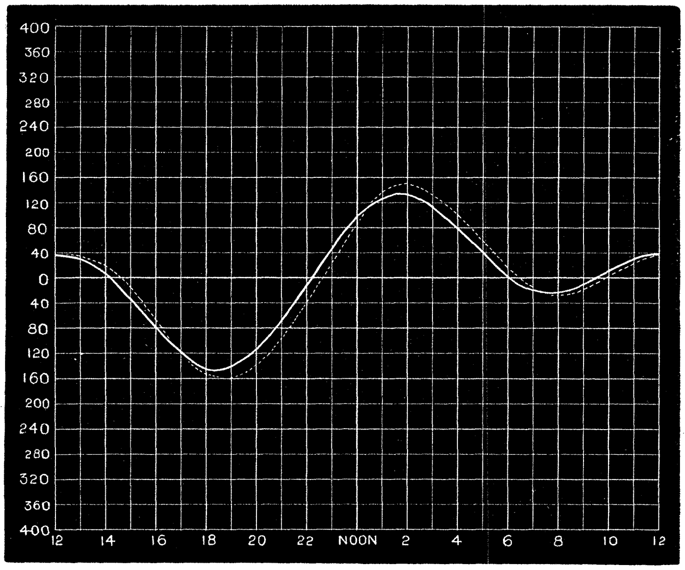
In order to show how far these numbers correctly represent the forces which have been made use of in their computation I have calculated backwards from the potential the force to geographical West. Table VIII. exhibits the results, showing by comparison the coefficients calculated from the formulæ for the potential with those obtained directly by observation. It will be seen that the agreement is satisfactory, except for the coefficients $a_3$ and $b_3$.

In figs. 1, 2, 3, and 4 the curves for the mean of the year are plotted. In these curves only the first four terms $a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t$ have been used. The numbers from which the curves are plotted are given in Table IX.

The curves are seen to be almost identical, and having, therefore, obtained an expression for the potential which correctly represents the observed West force, we may turn to the main object of this inquiry and calculate the vertical forces.

The calculation of the various components from the potential involves the knowledge of the differential coefficients of zonal harmonics and of the tesseral harmonics for the

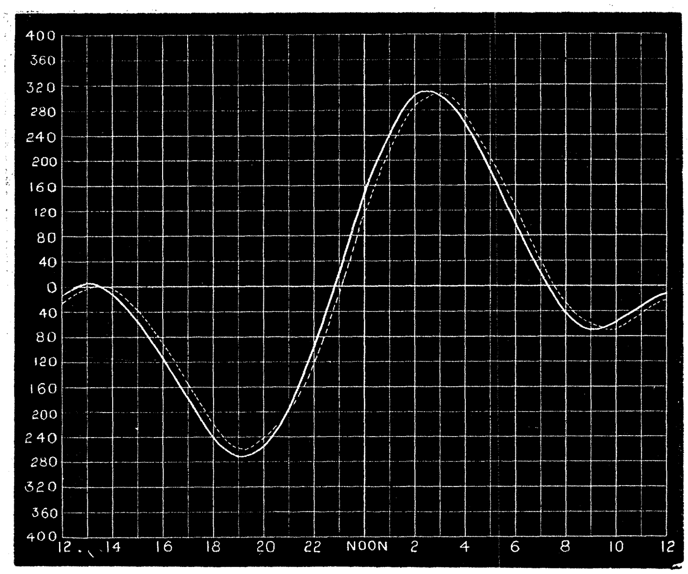
Fig. 1.




BOMBAY.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. $10^{-6}$. Observed curve, white line. Calculated curve, dotted line.

Fig. 2.




LISBON.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. $10^{-6}$. Observed curve, white line. Calculated curve, dotted line.




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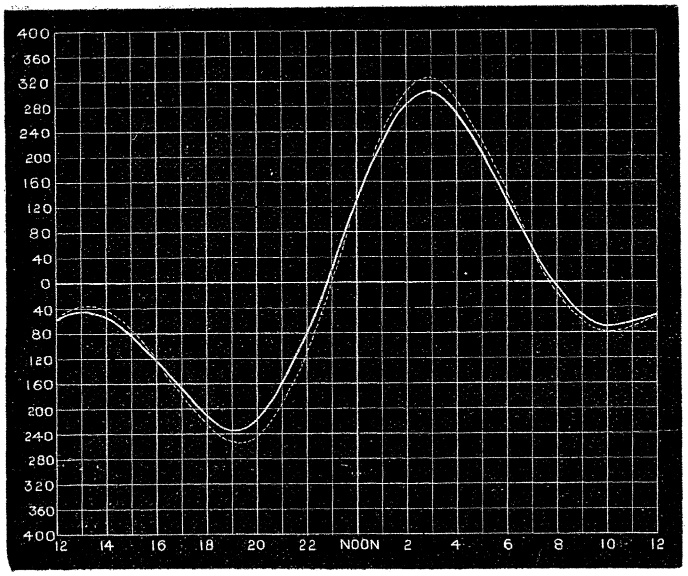
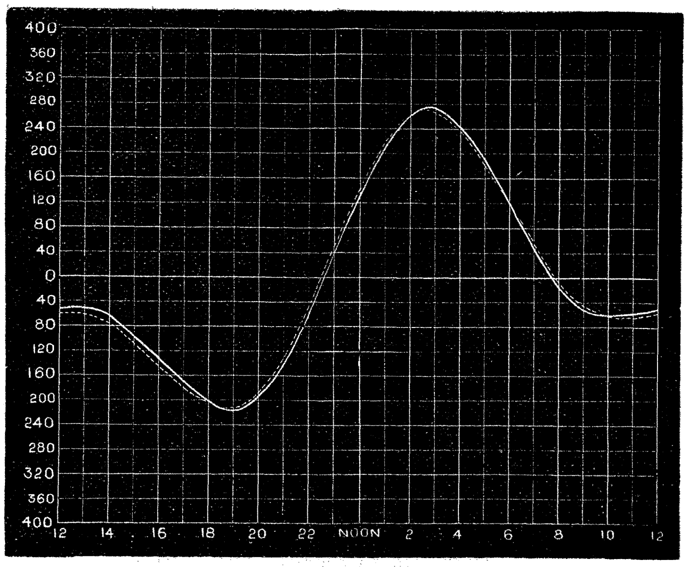
Fig. 3.




GREENWICH.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. $10^{-6}$.
Observed curve, white line. Calculated curve, dotted line.

Fig. 4.




ST. PETERSBURG.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. $10^{-6}$.
Observed curve, white line. Calculated curve, dotted line.

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four observing stations. The numbers which I have computed for this purpose are given in Table X.

In that table $T_p^9$ stands for

$$\frac{d^9 P_p}{d\mu^9} \sin^9 u,$$

where $\mu = \cos u$ and $P_p$ is the zonal harmonic of degree $p$.

TABLE VIII.—Comparison between observed and calculated coefficients of West force.

| | | $a_1$. | | $b_1$. | | $a_2$. | | $b_2$. | |
|---|---|---|---|---|---|---|---|---|---|
| | | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. |
| Bombay | Summer | + 58 | + 58 | + 116 | + 124 | + 132 | + 124 | + 60 | + 73 |
| | Winter | 0 | − 4 | + 32 | + 41 | + 4 | + 8 | + 23 | + 39 |
| | Mean | + 29 | + 27 | + 74 | + 82 | + 68 | + 66 | + 41 | + 56 |
| Lisbon | Summer | + 74 | + 88 | + 225 | + 232 | + 108 | + 90 | + 158 | + 172 |
| | Winter | + 87 | + 52 | + 119 | + 112 | + 31 | + 10 | + 104 | + 105 |
| | Mean | + 80 | + 70 | + 172 | + 172 | + 70 | + 50 | + 131 | + 138 |
| Greenwich | Summer | + 85 | + 91 | + 233 | + 255 | + 83 | + 85 | + 135 | + 149 |
| | Winter | + 105 | + 95 | + 115 | + 113 | + 1 | − 5 | + 87 | + 104 |
| | Mean | + 95 | + 93 | + 174 | + 184 | + 42 | + 40 | + 111 | + 127 |
| St. Petersburg | Summer | + 100 | + 79 | + 225 | + 229 | + 99 | + 92 | + 115 | + 100 |
| | Winter | + 92 | + 121 | + 96 | + 93 | − 15 | − 6 | + 69 | + 72 |
| | Mean | + 96 | + 100 | + 161 | + 161 | + 42 | + 43 | + 92 | + 87 |

| | | $a_3$. | | $b_3$. | | $a_4$. | | $b_4$. | |
|---|---|---|---|---|---|---|---|---|---|
| | | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. |
| Bombay | Mean of year | + 71 | + 19 | + 8 | − 24 | + 24 | + 20 | − 14 | − 12 |
| Lisbon | ” | + 63 | + 108 | + 57 | + 88 | + 24 | + 21 | + 8 | + 5 |
| Greenwich | ” | + 41 | + 82 | + 44 | + 74 | + 17 | + 31 | + 14 | + 21 |
| St. Petersburg | ” | − 12 | − 7 | − 8 | + 16 | − 18 | − 20 | − 1 | + 27 |

TABLE IX.—Comparison between calculated and observed variation of force to geographical West. The unit of force is C.G.S. $10^{-6}$. The first four terms only of the series $a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t$ have been taken into account.

| Astronomical time. | Bombay. | | Lisbon. | | Greenwich | | St. Petersburg. | |
|---|---|---|---|---|---|---|---|---|
| | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. |
| $a_1$ | + 29 | + 27 | + 80 | + 70 | + 95 | + 93 | + 96 | + 100 |
| $b_1$ | + 74 | + 82 | + 172 | + 172 | + 174 | + 184 | + 161 | + 161 |
| $a_2$ | + 68 | + 66 | + 70 | + 50 | + 42 | + 40 | + 42 | + 43 |
| $b_2$ | + 41 | + 56 | + 131 | + 138 | + 111 | + 127 | + 92 | + 87 |
| Hour. | | | | | | | | |
| 0 | + 97·0 | + 93·0 | + 150·0 | + 120·0 | + 137·0 | + 133·0 | + 138·0 | + 143·0 |
| 1 | + 126·6 | + 132·5 | + 247·9 | + 224·4 | + 228·8 | + 235·6 | + 216·8 | + 219·0 |
| 2 | + 131·6 | + 145·9 | + 303·7 | + 291·1 | + 286·4 | + 302·5 | + 264·3 | + 263·9 |
| 3 | + 113·8 | + 133·1 | + 309·2 | + 309·1 | + 301·2 | + 322·9 | + 273·7 | + 271·5 |
| 4 | + 80·1 | + 100·0 | + 267·4 | + 278·5 | + 273·3 | + 295·8 | + 246·1 | + 243·2 |
| 5 | + 40·6 | + 57·0 | + 191·8 | + 210·0 | + 211·8 | + 230·7 | + 190·0 | + 187·7 |
| 6 | + 6·0 | + 16·0 | + 102·0 | + 122·0 | + 132·0 | + 144·0 | + 119·0 | + 118·0 |
| 7 | − 15·4 | − 13·0 | + 19·4 | + 35·8 | + 51·6 | + 55·5 | + 48·2 | + 48·9 |
| 8 | − 19·9 | − 24·0 | − 39·4 | − 30·5 | − 13·9 | − 17·2 | − 9·3 | − 7·4 |
| 9 | − 9·2 | − 17·1 | − 66·0 | − 65·9 | − 55·2 | − 62·7 | − 46·1 | − 43·9 |
| 10 | + 10·4 | + 2·1 | − 61·7 | − 69·1 | − 70·4 | − 78·5 | − 61·3 | − 59·9 |
| 11 | + 29·6 | + 24·3 | − 37·7 | − 48·8 | − 65·8 | − 71·0 | − 60·6 | − 61·2 |
| 12 | + 39·0 | + 39·0 | − 10·0 | − 20·0 | − 53·0 | − 53·0 | − 54·0 | − 57·0 |
| 13 | + 32·2 | + 37·9 | + 4·3 | + 0·2 | − 45·0 | − 39·4 | − 52·0 | − 57·6 |
| 14 | + 7·4 | + 17·1 | − 6·9 | − 2·1 | − 52·2 | − 42·5 | − 62·9 | − 70·3 |
| 15 | − 31·8 | − 21·1 | − 47·2 | − 33·1 | − 79·2 | − 68·9 | − 89·7 | − 97·5 |
| 16 | − 77·1 | − 69·0 | − 110·6 | − 89·5 | − 123·1 | − 115·8 | − 128·7 | − 135·6 |
| 17 | − 117·4 | − 115·4 | − 182·0 | − 158·6 | − 173·6 | − 172·9 | − 170·8 | − 175·1 |
| 18 | − 142·0 | − 148·0 | − 242·0 | − 222·0 | − 216·0 | − 224·0 | − 203·0 | − 204·0 |
| 19 | − 143·4 | − 157·4 | − 271·6 | − 260·4 | − 235·4 | − 251·7 | − 213·0 | − 210·3 |
| 20 | − 119·1 | − 139·0 | − 257·4 | − 258·5 | − 220·3 | − 242·8 | − 192·1 | − 186·2 |
| 21 | − 72·8 | − 94·9 | − 196·0 | − 210·1 | − 166·8 | − 191·3 | − 137·9 | − 130·1 |
| 22 | − 13·4 | − 33·1 | − 95·1 | − 119·9 | − 79·8 | − 101·5 | − 56·1 | − 47·7 |
| 23 | + 47·2 | + 34·1 | + 27·9 | − 2·6 | + 27·6 | + 13·2 | + 41·4 | + 48·6 |



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TABLE X.—Numerical values for the zonal harmonics, their differential coefficients, and the functions $T_p^q$.

| | Bombay. | Lisbon. | Greenwich. | St. Petersburg. |
|---|---|---|---|---|
| $P_1$ . . . . . | + 0·3239 | + 0·6255 | + 0·7824 | + 0·8654 |
| $P_2$ . . . . . | − 0·3426 | + 0·0868 | + 0·4183 | + 0·6235 |
| $P_3$ . . . . . | − 0·4009 | − 0·3265 | + 0·0238 | + 0·3224 |
| $P_4$ . . . . . | + 0·0297 | − 0·4225 | − 0·2811 | + 0·0206 |
| $P_5$ . . . . . | + 0·3380 | − 0·2144 | − 0·4149 | − 0·2258 |
| $P_6$ . . . . . | + 0·1760 | + 0·1061 | − 0·3620 | − 0·3754 |
| $dP_2/d\mu$ . . . | + 0·9718 | + 1·8764 | + 2·3473 | + 2·5963 |
| $dP_3/d\mu$ . . . | − 0·7131 | + 1·4341 | + 3·0914 | + 4·1174 |
| $dP_4/d\mu$ . . . | − 1·8346 | − 0·4089 | + 2·5142 | + 4·8529 |
| $dP_5/d\mu$ . . . | − 0·4457 | − 2·3681 | + 0·5618 | + 4·3029 |
| $dP_6/d\mu$ . . . | + 1·8839 | − 2·7679 | − 2·0498 | + 2·3692 |
| $dP_7/d\mu$ . . . | + 1·8421 | − 0·9882 | − 4·1305 | − 0·5776 |
| $d^2P_2/d\mu^2$ . . | + 3·0000 | + 3·0000 | + 3·0000 | + 3·0000 |
| $d^2P_3/d\mu^2$ . . | + 4·8588 | + 9·3821 | + 11·7364 | + 12·9816 |
| $d^2P_4/d\mu^2$ . . | − 1·9916 | + 13·0386 | + 24·6401 | + 31·8221 |
| $d^2P_5/d\mu^2$ . . | − 11·6529 | + 5·7017 | + 34·3644 | + 56·6573 |
| $d^2P_6/d\mu^2$ . . | − 6·8945 | − 13·0104 | + 30·8207 | + 79·1540 |
| $d^2P_7/d\mu^2$ . . | + 12·8380 | − 30·2804 | + 7·7166 | + 87·4567 |
| $d^2P_8/d\mu^2$ . . | + 20·7369 | − 27·8334 | − 31·1374 | + 70·4907 |
| $d^3P_4/d\mu^3$ . . | + 34·011 | + 65·674 | + 82·155 | + 90·872 |
| $d^3P_6/d\mu^3$ . . | − 94·170 | + 128·393 | + 460·164 | + 714·102 |
| $d^3P_8/d\mu^3$ . . | + 98·399 | − 325·813 | + 575·912 | + 2025·953 |
| $d^4P_5/d\mu^3$ . . | +306·10 | + 591·07 | + 739·39 | + 817·84 |
| $d^4P_7/d\mu^3$ . . | −918·11 | +2260·17 | + 6721·52 | +10101·17 |
| $d^4P_9/d\mu^3$ . . | +754·68 | −3278·66 | +16512·03 | +44542·37 |
| $T_1^1$ . . . . . | + 0·9194 | + 1·4640 | + 1·4620 | + 1·3008 |
| $T_2^1$ . . . . . | − 0·6747 | + 1·1190 | + 1·9253 | + 2·0629 |
| $T_3^1$ . . . . . | − 1·7358 | − 0·3190 | + 1·5658 | + 2·4313 |
| $T_4^1$ . . . . . | − 0·4217 | − 1·8477 | + 0·3498 | + 2·1558 |
| $T_5^1$ . . . . . | + 1·7825 | − 2·1596 | − 1·2766 | + 1·1870 |
| $T_6^1$ . . . . . | + 1·7429 | − 0·7710 | − 2·5725 | − 0·2894 |
| $T_7^1$ | | | | |
| $T_2^2$ . . . . . | + 2·685 | + 1·826 | + 1·163 | + 0·753 |
| $T_3^2$ . . . . . | + 4·349 | + 5·712 | + 4·551 | + 3·258 |
| $T_4^2$ . . . . . | − 1·783 | + 7·938 | + 9·556 | + 7·987 |
| $T_5^2$ . . . . . | − 10·430 | + 3·471 | + 13·327 | + 14·221 |
| $T_6^2$ . . . . . | − 6·171 | − 7·921 | + 11·953 | + 19·868 |
| $T_7^2$ . . . . . | + 11·491 | − 18·435 | + 2·993 | + 21·952 |
| $T_8^2$ . . . . . | + 18·561 | − 16·945 | − 12·075 | + 17·693 |
| $T_4^3$ . . . . . | + 28·801 | + 31·195 | + 19·841 | + 11·427 |
| $T_6^3$ . . . . . | − 79·745 | + 60·985 | + 111·130 | + 89·801 |
| $T_8^3$ . . . . . | + 83·326 | − 154·760 | + 139·090 | + 254·770 |
| $T_5^4$ . . . . . | +245·23 | + 219·06 | + 111·20 | + 51·525 |
| $T_7^4$ . . . . . | −735·55 | + 837·66 | + 1010·90 | + 636·390 |
| $T_9^4$ . . . . . | +604·62 | −1215·10 | + 2483·40 | + 2806·300 |

## IV. *Comparison of Observed and Calculated Vertical Forces.*

The complete potential can be written down from the value at its surface in the usual manner, either on the supposition that the potential is zero at an infinite distance, or that it is zero at the centre of the Earth, the first supposition corresponding to the hypothesis that the seat of the magnetic variation is outside the Earth. If $Y_n$ is a surface harmonic of degree $n$ occurring in the expansion of $V/a$, the solid harmonic will either be $Y_n(r/a)^n$ or $Y_n(r/a)^{-(n+1)}$. The vertical force is given by $\partial V/\partial r$, as the force is considered positive when it acts downwards. At the surface, therefore, we have a term for the vertical force which is either $nY_n$ or $-(n+1)Y_n$.

Before we proceed to discuss the comparison between the observed and calculated values of the vertical force, a few words are necessary regarding the available observations.

The only station for which we have complete records for 1870 is Lisbon. It is therefore impossible to obtain a satisfactory series for the vertical force which would give us, if our information was more complete, directly the two terms, one due to outside, the other due to inside, effects. But I shall show that even from the existing data we can draw important conclusions.

At Bombay no vertical force determinations are published, as far as I know, before 1873, when the magnetograph came into operation; but we have complete records between 1873 and 1878. During these years the general type of the vertical force remained practically the same, only the range varying. Figs. 5 and 6, for instance,

Fig. 5.

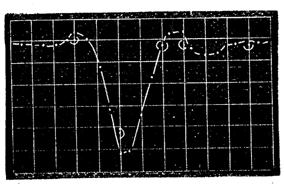
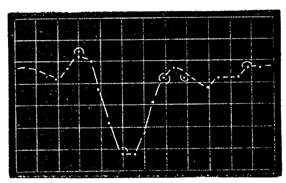


Fig. 6.



are tracings of the curves given by Mr. CHAMBERS in the years 1873 and 1877, the two years which differ most in range. The other years show curves varying between these two extremes. As far as the general form of the curve is concerned, we cannot go far wrong, therefore, if we make use of the 1873 observations, especially as the horizontal components show no marked difference (except as regards range) in 1873 and 1870.

Similar remarks apply to Greenwich. Although published records exist for 1870, there is an uncertainty about the temperature correction, which makes the vertical force observations previous to 1882 useless for our present purpose.

Table XI. will give an idea of the changes which the vertical force variation is subject to in various years. The numbers are copied from the published records of the Greenwich Observatory.

TABLE XI.

Coefficients in the series

$$V_t = c_1 \sin(t + \alpha) + c_2 \sin(2t + \beta) + c_3 \sin(3t + \gamma) + c_4 \sin(4t + \gamma)$$

where changes in $V_t$ represent changes in the vertical force, the unit being ·00001 of the whole vertical force; $t$ being the time from midnight.

| | $c_1$. | $\alpha$. | $c_2$. | $\beta$. | $c_3$. | $\gamma$. | $c_4$. | $\delta$. |
|---|---|---|---|---|---|---|---|---|
| | | ° | | ° | | ° | | ° |
| 1883 | 14·3 | 148·13 | 13·1 | 266·58 | 5·3 | 89·60 | 1·3 | 293·20 |
| 1884 | 14·8 | 139·33 | 11·7 | 272·00 | 5·5 | 95·52 | 2·1 | 289·49 |
| 1885 | 13·0 | 137·50 | 11·7 | 265·35 | 5·1 | 83·28 | 1·5 | 281·04 |
| 1886 | 12·5 | 160·58 | 11·9 | 268·38 | 4·0 | 94·22 | 1·2 | 297·50 |

The principal discrepancy here occurs in the angle $\alpha$, which would give a difference of phase between 1885 and 1886 of about 23°, or about an hour and a half; the angle $\alpha$ differs much at Greenwich during different months; the phases of the other terms show practically no difference. In comparing the observed and calculated values, I have taken the year 1884, as during that year the range of the declination needle was greatest, and corresponded most nearly to that of 1870.

At St. Petersburg, also, the results for 1870 are not corrected for temperature, and there is reason to believe that this has affected the observed type considerably. I have therefore compared the results of calculation with observation, also, for the year 1878, in which year the temperature corrections have been taken into account.

We must remember, then, in the comparison between the observed and calculated values, that the greatest weight must attach to the Lisbon observations, and that comparatively little value is to be given to St. Petersburg, as I have no records at my disposal which would enable me to judge how far the type of vertical force there varies from year to year.

Tables XII. and XIII. give the values of the coefficients $a_1$, $b_1$, $a_2$, $b_2$, &c., as calculated on the hypothesis that the disturbing force comes from the inside of the Earth. Tables XIV. and XV. give their values calculated on the hypothesis that the disturbing force is outside. In both cases the observed numbers are given for comparison. I have calculated the values for Lisbon separately for the winter and for the summer months; in the other cases the values for the mean of the year only have been taken. The years attached to the observing stations refer to the date of the observations; the calculated values all belong to 1870.

TABLE XII.—Comparison between the observed and calculated coefficients for the vertical force, $a_1$, $b_1$, $a_2$, $b_2$ on the hypothesis that the disturbing force is inside the Earth.

| | $a_1$. | | $b_1$. | | $a_2$. | | $b_2$. | |
|---|---|---|---|---|---|---|---|---|
| | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. |
| Bombay . . . . . . . | + 219 | − 42 | − 57 | + 9 | + 79 | − 21 | − 152 | + 28 |
| Lisbon, summer . . . | + 580 | − 196 | − 206 | + 66 | + 381 | − 201 | − 87 | + 26 |
| " winter . . . . | + 336 | − 135 | − 149 | + 55 | + 277 | − 103 | − 19 | + 1 |
| Greenwich, 1884 . . . | + 350 | − 42 | − 190 | + 49 | + 139 | − 51 | − 34 | + 2 |
| St. Petersburg, 1870. . | + 177 | + 114 | − 154 | + 125 | + 62 | − 59 | − 45 | − 38 |
| " 1878. . | | − 8 | .. | + 29 | .. | − 24 | .. | − 2 |

TABLE XIII.—Comparison between the observed and calculated coefficients $a_3$, $b_3$, $a_4$, $b_4$ for the vertical force on the hypothesis that the disturbing force is inside the Earth.

| | $a_3$. | | $b_3$. | | $a_4$. | | $b_4$. | |
|---|---|---|---|---|---|---|---|---|
| | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. |
| Bombay . . . . . . . | − 73·4 | − 10 | − 42·0 | + 35 | − 20·4 | + 3 | − 29·0 | + 16 |
| Lisbon . . . . . . . | +156·1 | − 81 | −171·4 | + 12 | + 9·2 | − 21 | − 22·1 | + 11 |
| Greenwich . . . . . . | + 85·5 | − 24 | − 88·0 | + 3 | + 28·1 | − 8 | − 38·2 | + 3 |
| St. Petersburg . . . . | + 12·0 | − 12 | − 17·4 | − 8 | + 22·3 | − 18 | − 31·3 | − 1 |

TABLE XIV.—Comparison between the observed and calculated coefficients $a_1$, $b_1$, $a_2$, $b_2$ on the assumption that the disturbing force is outside the Earth.

| | $a_1$. | | $b_1$. | | $a_2$. | | $b_2$. | |
|---|---|---|---|---|---|---|---|---|
| | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. |
| Bombay . . . . . . . | − 141 | − 42 | + 31 | + 9 | − 53 | − 21 | + 121 | + 28 |
| Lisbon, summer . . . | − 398 | − 196 | + 137 | + 66 | − 315 | − 201 | + 52 | + 26 |
| " winter . . . . | − 248 | − 135 | + 108 | + 55 | − 235 | − 103 | + 15 | + 1 |
| Greenwich, 1884 . . . | − 235 | − 42 | + 132 | + 49 | − 110 | − 51 | + 21 | + 2 |
| St. Petersburg, 1870 . . | − 97 | + 114 | + 104 | + 125 | − 40 | − 59 | + 35 | − 38 |
| " 1878 . . | .. | − 8 | .. | + 29 | .. | − 24 | .. | − 2 |

TABLE XV.—Comparison between the observed and calculated coefficients $a_3$, $b_3$, $a_4$, $b_4$ on the assumption that the disturbing force is outside the Earth.

| | $a_3$. | | $b_3$. | | $a_4$. | | $b_4$. | |
|---|---|---|---|---|---|---|---|---|
| | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. | Calculated. | Observed. |
| Bombay | + 66 | − 10 | + 22 | + 35 | + 18 | + 3 | + 24 | + 16 |
| Lisbon | − 134 | − 81 | + 125 | + 12 | − 8 | − 21 | + 18 | + 11 |
| Greenwich | − 70 | − 24 | + 38 | + 3 | − 25 | − 8 | + 33 | + 3 |
| St. Petersburg | − 7 | − 12 | + 16 | − 8 | − 20 | − 18 | + 27 | − 1 |

Confining ourselves in the first place to the first four coefficients, we find that out of twenty coefficients eighteen have the wrong sign on the hypothesis of an internal cause, while only two have the wrong sign on the hypothesis of an external cause, and those two belong to St. Petersburg, to which station, as was pointed out, we cannot attach much value. If instead of the numbers given for 1870 we take the numbers given at the same station for 1878, the agreement becomes better, even for St. Petersburg. The coefficients $a_3$, $b_3$, $a_4$, $b_4$ are of course more uncertain; but even here the evidence is strongly in favour of the external cause. Out of twenty coefficients seventeen agree in sign with that hypothesis.

A better comparison can perhaps be obtained in a different way: the two terms

$$a_n \cos nt + b_n \sin nt$$

can be written

$$r_n \cos n (t - t_n),$$

where $r_n$ is the amplitude of the oscillation, and $t_n$ the time at which the maximum elongations take place.

Tables XVI. and XVII. contain the results for $t_n$, and from these tables I think it will clearly appear that the phase of the vertical force completely agrees with the assumption of an *external* cause and completely disagrees with the assumption of an *internal* cause.

For Lisbon, our principal station of comparison, the phase in Table XVI. agrees for both the diurnal and semi-diurnal variation within four minutes of time. For Bombay the diurnal variation agrees within three minutes, and the semi-diurnal variation within 36 minutes. For Greenwich, the semi-diurnal variation, which we have seen differs little from year to year, agrees closely, while the diurnal variation shows a greater difference. In all these cases the phases, as calculated in Table XV., are in as great a disagreement as possible. St. Petersburg gives less decisive results, but they still go in the same direction, especially if we take the observations of 1878 to represent the type of vertical variation.

The amplitudes of the variation are given in Table XVIII. where the second column gives the calculated values on the hypothesis which we must reject, while the third column gives the same numbers calculated on the assumption of an external cause.

The calculated numbers are much larger than the observed ones, which is a fact requiring explanation. But, the range of vertical force differing from year to year, we must confine ourselves in our reasoning principally to the Lisbon observations.

The changes in the range of different years, as far as we have any observations, are not so considerable, however, as to account for the discrepancies at Bombay or Greenwich, and we must conclude, I believe, that also for those stations a considerably smaller range is obtained in the observed than in the calculated forces.

TABLE XVI.—Observed and calculated values of the coefficients $t_1$ and $t_2$ of vertical force, when expressed in the form $r_1 \cos(t - t_1) + r_2 \cos 2(t - t_2)$ on the supposition that the disturbing force is inside the Earth.

| | $t_1$. | | | $t_2$. | | |
|---|---|---|---|---|---|---|
| | Calculated. | Observed. | Difference. | Calculated. | Observed. | Difference. |
| | h. m. | h. m. | h. m. | h. m. | h. m. | h. m. |
| Bombay | 23 02 | 11 13 | + 11 49 | 9 55 | 4 23 | + 5 32 |
| Lisbon | 22 35 | 10 40 | + 11 58 | 11 42 | 5 50 | + 5 52 |
| Greenwich | 22 06 | 8 42 | − 11 57 | 11 32 | 5 56 | + 5 36 |
| St. Petersburg, 1870 | 21 16 | 3 10 | − 5 54 | 10 48 | 7 05 | + 3 43 |
| „ 1878 | .. | 7 05 | − 9 49 | .. | 6 12 | + 4 36 |

TABLE XVII.—Observed and calculated values of the coefficients $t_1$ and $t_2$ when expressed in the form $r_1 \cos(t - t_1) + r_2 \cos 2(t - t_2)$ on the supposition that the disturbing force is outside the Earth.

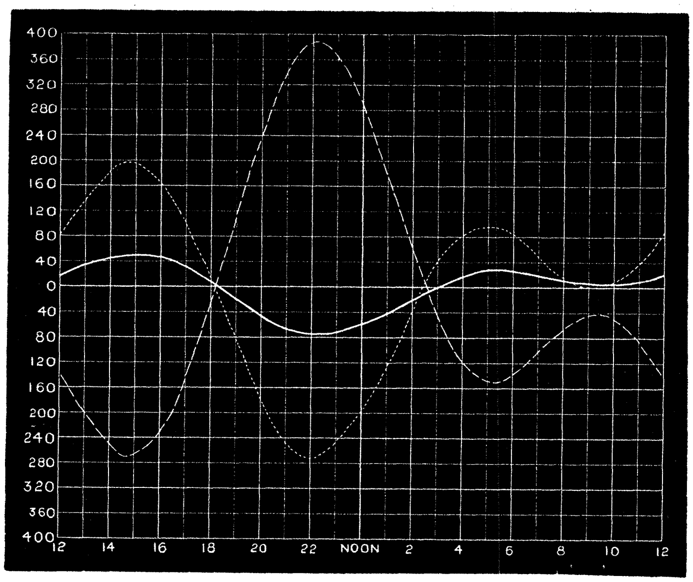
| | $t_1$. | | | $t_2$. | | |
|---|---|---|---|---|---|---|
| | Calculated. | Observed. | Difference. | Calculated. | Observed. | Difference. |
| | h. m. | h. m. | h. m. | h. m. | h. m. | h. m. |
| Bombay | 11 10 | 11 13 | − 0 03 | 3 47 | 4 23 | − 0 36 |
| Lisbon | 10 37 | 10 40 | − 0 03 | 5 46 | 5 50 | − 0 04 |
| Greenwich | 10 03 | 8 42 | + 1 21 | 5 38 | 5 56 | − 0 18 |
| St. Petersburg, 1870 | 8 52 | 3 10 | + 5 42 | 4 38 | 7 05 | − 2 27 |
| „ 1878 | .. | 7 05 | − 1 47 | .. | 6 12 | − 1 34 |

TABLE XVIII.—Observed and calculated values of $r_1$ and $r_2$ in the expression $r_1 \cos(t - t_1) + r_2 \cos 2(t - t_2)$ for vertical force.

| | $r_1$. | | | $r_2$. | | |
|---|---|---|---|---|---|---|
| | Calculated from inside. | Calculated from outside. | Observed. | Calculated from inside. | Calculated from outside. | Observed. |
| Bombay . . . . . . . | 226 | 144 | 43 | 171 | 132 | 35 |
| Lisbon . . . . . . . | 491 | 346 | 176 | 333 | 277 | 153 |
| Greenwich . . . . . . | 398 | 269 | 65 | 143 | 112 | 51 |
| St. Petersburg, 1870 . . | 235 | 142 | 169 | 77 | 53 | 71 |
| „ 1878 . . | .. | .. | 30 | .. | .. | 24 |

The agreement between the calculated and observed curves for vertical force is best seen from the graphical representation given in figs. 7, 8, 9, 10. In the curves, the diurnal and semidiurnal variation only have been taken into account. It is seen how at Bombay, Lisbon, and Greenwich the observed curves are almost identical in shape with the curves calculated on the hypothesis of an external cause, if the range is reduced in a proper ratio. The disagreement with the curves calculated on the other alternative is complete, a maximum occurring where a minimum should occur, and *vice versâ*. At St. Petersburg, although the agreement is much worse, the 1878 curve, which has been corrected for temperature, follows pretty closely the shape of the calculated curve. The numbers from which the curves have been drawn are given in Table XIX.

Fig. 7.

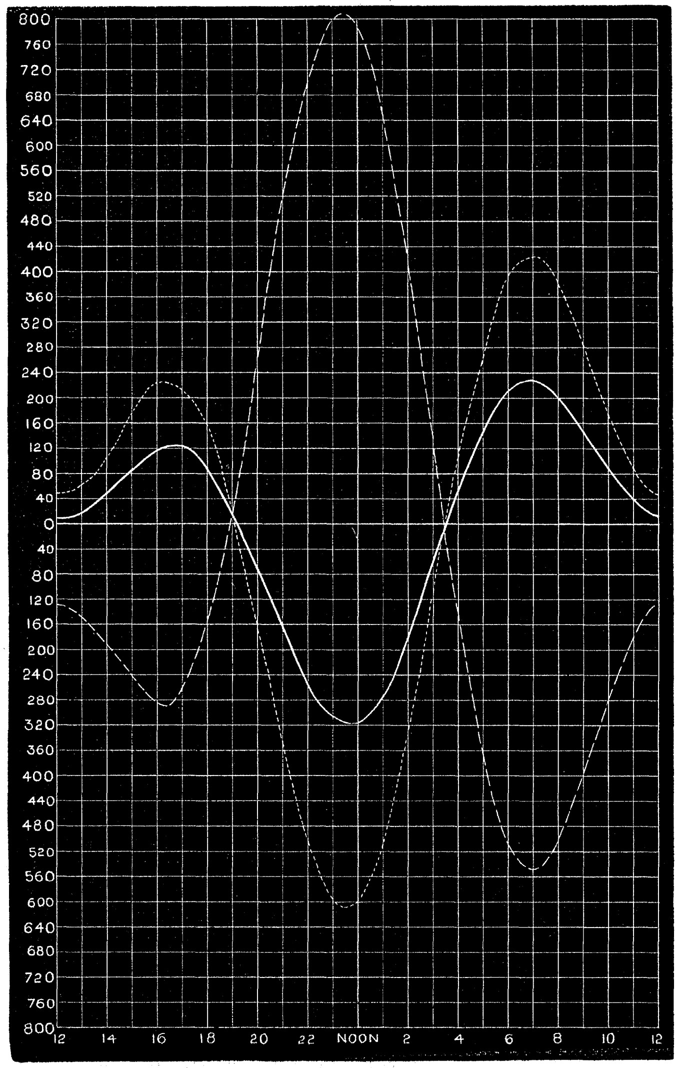



BOMBAY.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. $10^{-6}$.

Observed curve, white line. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.

Fig. 8.




LISBON.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. $10^{-6}$.

Observed curve, white line. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.



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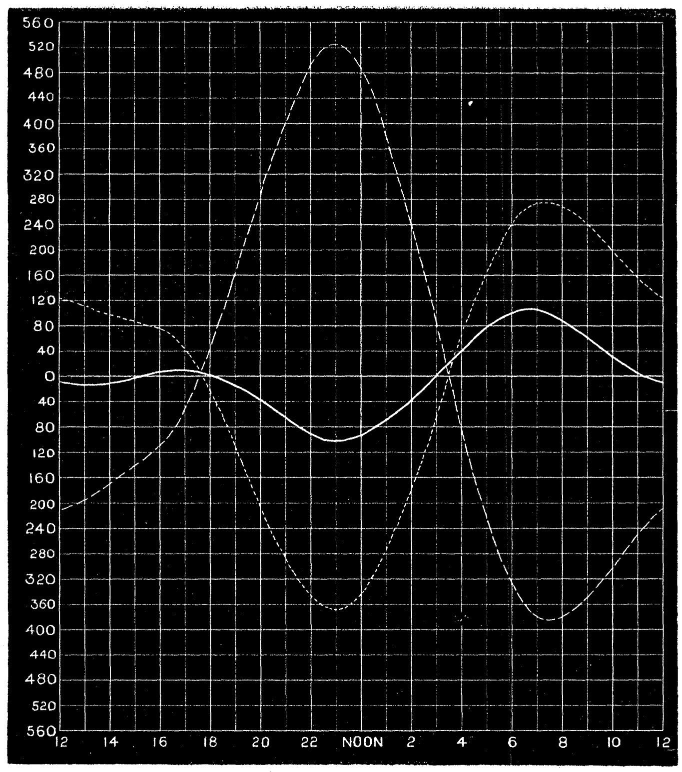
Fig. 9.

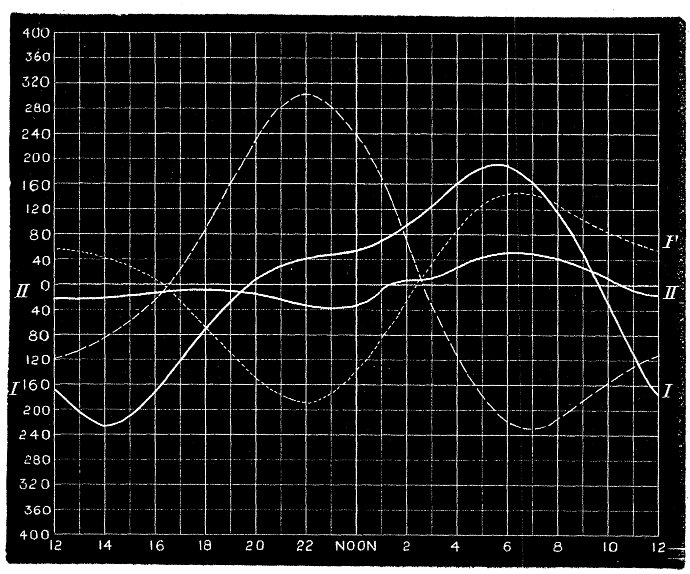



GREENWICH.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. $10^{-6}$.

Observed curve, white line. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.

Fig. 10.




ST. PETERSBURG.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. $10^{-6}$.

Observed curves, white line—I. for 1870, II. for 1878. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.

TABLE XIX.—Comparison between calculated and observed variation of vertical force. The unit of force is C.G.S. $10^{-6}$. The first four terms of the series $a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t$ have been taken into account.

| Astronomical time. | Bombay. | | | Lisbon. | | | Greenwich. | | | St. Petersburg. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Observed. | Calculated from outside. | Calculated from inside. | Observed. | Calculated from outside. | Calculated from inside. | Observed. | Calculated from outside. | Calculated from inside. | Observed 1870. | Calculated from outside. | Calculated from inside. | Observed 1878. |
| | $a_1$ −42 | −141 | +219 | $a_1$ −165 | −323 | +458 | $a_1$ −42 | −235 | +350 | +114 | − 97 | +177 | − 8 |
| | $b_1$ + 9 | + 31 | − 57 | $b_1$ + 60 | +122 | −177 | $b_1$ +49 | +132 | −190 | +125 | +104 | −154 | +29 |
| | $a_2$ −21 | − 53 | + 79 | $a_2$ −152 | −275 | +329 | $a_2$ −51 | −110 | +139 | − 59 | − 40 | + 62 | −24 |
| | $b_2$ +28 | +121 | −152 | $b_2$ + 13 | + 34 | − 53 | $b_2$ + 2 | + 21 | − 34 | − 38 | + 35 | − 45 | − 2 |
| Hour. | | | | | | | | | | | | | |
| 0 | −63·0 | −194·0 | +298·0 | −317·0 | −598·0 | +787·0 | − 93·0 | −345·0 | +489·0 | + 55·0 | −137·0 | +239·0 | −32·0 |
| 1 | −42·5 | −113·6 | +189·2 | −269·0 | −501·6 | +655·0 | − 71·1 | −277·6 | +392·3 | + 72·4 | − 83·9 | +162·3 | −22·0 |
| 2 | −18·2 | − 28·3 | + 69·1 | −177·6 | −326·8 | +426·7 | − 35·7 | −174·3 | +248·2 | + 98·8 | − 21·7 | + 68·3 | − 6·1 |
| 3 | + 4·7 | + 43·2 | − 37·5 | − 61·3 | −108·1 | +145·7 | + 6·9 | − 51·8 | + 79·2 | +131·0 | + 39·9 | − 28·8 | +12·8 |
| 4 | +21·5 | + 87·6 | −111·0 | + 56·8 | +111·1 | −134·7 | + 48·6 | + 70·0 | − 88·4 | +161·9 | + 91·9 | −114·9 | +31·4 |
| 5 | +30·0 | + 99·8 | −142·8 | +153·4 | +289·4 | −363·8 | + 81·6 | +172·4 | −230·2 | +182·4 | +127·5 | −179·2 | +45·7 |
| 6 | +30·0 | + 84·0 | −136·0 | +212·0 | +397·0 | −506·0 | +100·0 | +242·0 | −329·0 | +184·0 | +144·0 | −216·0 | +53·0 |
| 7 | +23·8 | + 51·8 | −104·2 | +225·8 | +422·8 | −548·0 | +101·4 | +273·2 | −377·6 | +161·4 | +142·7 | −225·8 | +51·9 |
| 8 | +15·1 | + 19·0 | − 66·8 | +199·2 | +375·3 | −500·9 | + 87·2 | +268·6 | −379·6 | +113·7 | +128·3 | −213·9 | +42·8 |
| 9 | + 8·1 | + 0·6 | − 43·1 | +146·1 | +280·7 | −395·9 | + 62·3 | +238·4 | −347·8 | + 45·8 | +107·1 | −189·0 | +28·2 |
| 10 | + 6·2 | + 6·3 | − 47·1 | + 85·6 | +173·8 | −274·7 | + 33·7 | +196·3 | −299·2 | − 32·8 | + 85·7 | −160·3 | +11·1 |
| 11 | +10·7 | + 37·8 | − 82·0 | + 36·8 | + 88·4 | −176·8 | + 8·1 | +155·4 | −249·9 | −109·8 | + 68·5 | −134·7 | − 4·6 |
| 12 | +21·0 | + 88·0 | −140·0 | + 13·0 | + 48·0 | −129·0 | − 9·0 | +125·0 | −211·0 | −173·0 | + 57·0 | −115·0 | −16·0 |
| 13 | +34·1 | +142·8 | −204·4 | + 18·8 | + 59·2 | −138·2 | − 15·3 | +108·0 | −185·5 | −212·6 | + 49·7 | − 99·9 | −21·6 |
| 14 | +45·6 | +184·9 | −253·3 | + 48·2 | +110·6 | −189·5 | − 11·9 | +100·7 | −168·0 | −223·6 | + 42·3 | − 84·3 | −21·3 |
| 15 | +51·3 | +198·8 | −266·5 | + 87·3 | +176·1 | −251·7 | − 2·9 | + 93·8 | −147·2 | −207·0 | + 30·1 | − 61·2 | −16·8 |
| 16 | +47·9 | +175·0 | −231·2 | +117·8 | +222·7 | −286·1 | + 5·8 | + 76·4 | −109·4 | −168·7 | + 8·7 | − 25·1 | −10·8 |
| 17 | +34·4 | +113·0 | −146·0 | +122·8 | +221·0 | −259·0 | + 8·8 | + 39·2 | − 44·6 | −118·2 | − 23·3 | + 26·8 | − 6·1 |
| 18 | +12·0 | + 22·0 | − 22·0 | + 92·0 | +153·0 | −152·0 | + 2·0 | − 22·0 | + 51·0 | − 66·0 | − 64·0 | + 92·0 | − 5·0 |
| 19 | −15·4 | − 81·0 | +119·4 | + 24·4 | + 19·6 | + 31·2 | − 15·0 | −103·6 | +170·8 | − 21·2 | −108·5 | +163·4 | − 8·3 |
| 20 | −42·5 | −175·6 | +251·0 | − 69·8 | −159·1 | +263·7 | − 39·6 | −195·0 | +299·4 | + 11·1 | −148·9 | +229·9 | −15·4 |
| 21 | −64·1 | −242·6 | +347·1 | −172·1 | −348·7 | +501·9 | − 66·3 | −280·4 | +415·8 | + 30·2 | −177·1 | +279·0 | −24·2 |
| 22 | −75·6 | −268·9 | +389·3 | −260·2 | −507·6 | +695·5 | − 88·1 | −342·7 | +497·0 | + 39·6 | −186·3 | +300·3 | −31·7 |
| 23 | −75·1 | −250·6 | +370·8 | −313·0 | −598·8 | +799·6 | − 98·5 | −367·0 | +524·7 | + 45·6 | −172·7 | +287·1 | −35·0 |

If we, then, take it as proved that the primary cause of the variation comes to us from outside the Earth's surface, we are led to consider that a varying magnetic potential must cause induced currents within the Earth, if that body is a sufficiently good conductor. These induced currents might be the cause of the apparent reduction in amplitude. As my colleague, Professor LAMB, had given considerable attention to the problem of currents inside a conducting sphere, I consulted him, and he gave me the formulæ by means of which the induced currents can be calculated. His investigation is added as an Appendix to this paper.

## V. *Discussion of effects due to Currents Induced in the Inside of the Earth.*

I shall assume, then, for the present, that there is a periodic magnetic disturbance having its cause outside the Earth, and being probably due to electric currents in our atmosphere. Currents will be induced within the Earth, and we must now discuss what the effect of these currents will be, and whether they will account for the reduction in amplitude of the vertical forces which the observations show.

The varying potential can be expressed as a sum of terms of the form

$$\Omega_n \cos(pt + \lambda),$$

where $\Omega$ is a solid harmonic of degree $n$. Professor LAMB's formulæ allow us to calculate for each value of $n$, and for each value of $p$, the magnetic effect due to the induced currents, on the supposition that the specific resistance $\rho$ of the Earth is uniform. The forces due to these currents will have a different amplitude and a different phase from the original forces, and it is the resultant effect which we observe in the diurnal variations. The general effect will be to increase the horizontal components and to diminish the vertical component. The difference of phase will be the same for all components, provided we give a different sign to the amplitude of the vertical components of the inside and outside currents respectively. Otherwise the difference of phase of the vertical component will be greater by two right angles than the difference of phase of the horizontal components.

If one of the horizontal forces and the vertical force due to the solid harmonic of positive degree $n$ are written

$$a \cos(pt + \lambda) \quad \text{and} \quad b \cos(pt + \lambda + \epsilon),$$

the corresponding components due to the induced potential of negative degree will be of the form

$$c'a \cos(pt + \lambda + \alpha) \quad \text{and} \quad cb \cos(pt + \lambda + \epsilon + \alpha).$$

Table XX. gives the coefficients $c'$, $c$, and $\alpha$ for given values of the specific resistance $\rho$, if $n = 2$. The value of $\delta$ has the same meaning as in Professor LAMB's paper, and is connected with $\rho$ by the equation

$$\rho\delta = 4\pi p R^2$$

where $p$ is equal to $2\pi m/T$, and $m$ is equal to 1 for the diurnal variation, and equal to 2 for the semidiurnal variation.

Table XXI. gives the same quantities for $n = 4$.

An example may render the use of the Table more intelligible. Let the Earth, for instance, have a uniform specific resistance, which in C.G.S. units is $1·23 \times 10^{13}$, and consider that term of the magnetic potential which, on the surface of the Earth, has the form

$$A \sin u \cos u \cos \frac{2\pi t}{T},$$

where T is the time of the revolution of the Earth. This term alone represents fairly well the characteristic features of the diurnal variation. Here, as $n = 2$, we may use Table XX. The value of $\delta$ corresponding to the assumed resistance is 30, and we find $c = -·5$ approximately, which means that the vertical force due to the induced currents has half the amplitude of the vertical force due to the primary currents; also, the difference in phase is 41°; as the sign of the amplitude is changed, this means that the *minimum* of the vertical force due to induced currents takes place not quite three hours after the *maximum* of the corresponding primary force. Similarly the horizontal force due to the inside current is only one-third of the corresponding horizontal force due to the outside potential, and here the *maximum* due to the secondary currents takes place nearly three hours after the *maximum* due to the secondary currents. To get similar numbers for the semidiurnal variation we should have to put $m = 2$, and find $c$, $c'$, and $\alpha$ for $\delta = 60$, because $\rho = 1·23 \times 10^{13} = 2 \times 6·15 \times 10^{12}$, and looking up $6·15 \times 10^{12}$ in the last column we should find the corresponding number in the first column to be 60.

TABLE XX.—Comparison of the magnetic forces due to a system of varying electric currents outside the Earth, with the forces due to the currents induced inside the Earth. $n = 2$.

| $\delta$. | $c$. Ratio of normal forces due to secondary and primary variation. | $c'$. Ratio of tangential forces due to secondary and primary variation. | $\alpha$. Difference of phase. | Corresponding value of $\rho$. |
|---|---|---|---|---|
| 1 | − ·02854 | + ·01903 | 87° 27½′ | $3·70 \times 10^{14} \times m$. |
| 2 | ·05688 | ·03792 | 84 55½ | $1·85 \times 10^{14}$ |
| 3 | ·08484 | ·05656 | 82 25 | $1·23 \times 10^{14}$ |
| 4 | ·11226 | ·07484 | 79 57 | $9·25 \times 10^{13}$ |
| 5 | ·13895 | ·09263 | 77 32½ | $7·40 \times 10^{13}$ |
| 6 | ·16479 | ·10986 | 75 10½ | $6·17 \times 10^{13}$ |
| 7 | ·18968 | ·12645 | 72 53 | $5·29 \times 10^{13}$ |
| 8 | ·21354 | ·14236 | 70 41 | $4·63 \times 10^{13}$ |
| 9 | ·23633 | ·15755 | 68 32 | $4·11 \times 10^{13}$ |
| 10 | ·25799 | ·17199 | 66 29 | $3·70 \times 10^{13}$ |
| 20 | ·41771 | ·27847 | 50 32 | $1·85 \times 10^{13}$ |
| 30 | ·50520 | ·33680 | 41 03 | $1·23 \times 10^{13}$ |
| 40 | ·56158 | ·37439 | 35 11 | $9·25 \times 10^{12}$ |
| 50 | ·59864 | ·39909 | 31 11 | $7·40 \times 10^{12}$ |
| 100 | ·69949 | ·46632 | 21 33 | $3·70 \times 10^{12}$ |

TABLE XXI.—Comparison of the magnetic force due to a varying potential represented by a solid harmonic of degree 4 with the corresponding forces due to currents induced inside the Earth.

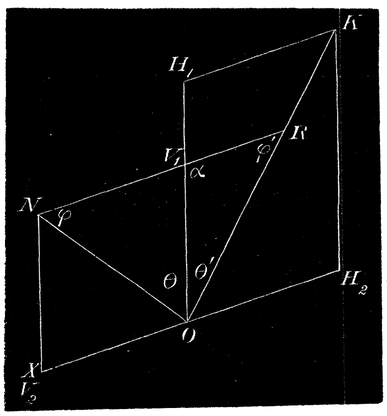
| $\delta$. | $c$. Ratio of normal forces due to secondary and primary variation. | $c'$. Ratio of tangential forces due to secondary and primary variation. | $\alpha$. Difference of phase. | Corresponding value of $\rho$. |
|---|---|---|---|---|
| 1 | $-\cdot0101$ | $+\cdot0081$ | 89° 01′ | $3\cdot70 \times 10^{14} \times m$. |
| 10 | ·0992 | ·0794 | 80 21 | $3\cdot70 \times 10^{13}$ |
| 100 | ·5149 | ·4119 | 37 50 | $3\cdot70 \times 10^{12}$ |

Tables XX. and XXI. cannot, however, be used to compare our calculated and observed results, but form only an intermediate step.

We observe on the Earth the resultant of the outside and inside effect, and we have calculated the vertical force on the assumption that the whole horizontal force is due to outside effect.

In fig. 11, let $OH_1$ represent that part of the horizontal force which is due to the

Fig. 11.




outside effect, and $OH_2$ the corresponding force in phase and amplitude which is due to the induced effect. The observed horizontal force will be the resultant OK. Let $OV_1$ be the magnitude of the vertical force due to the outside currents, and $OV_2$ the vertical force in phase and amplitude due to the induced effect. The observed vertical force will be ON. If we calculate the vertical force on the assumption that the resultant OK is due entirely to the outside, we should obtain a force OR, such that $OR : OK :: OV_1 : OH_1$. Our calculated value of vertical force will be OR, and our observed value ON. We require the ratio of lengths of these lines and the angle between them.

In the triangle ONR we know, from Table XX., the ratios

$$V_1R : OV_1 = H_1K : OH_1 = OH_2 : OH_1 = c',$$
$$V_1N : OV_1 = OV_2 : OV_1 = c,$$

also the angle

$$NV_1O = V_1OH_2 = \alpha.$$

Write

$$NOV_1 = \theta;\ ROV_1 = \theta';\ ONV_1 = \phi;\ ORV_1 = \phi'.$$

The triangles $OV_1N$ and $OV_1R$ give us the equations

$$\tan \tfrac{1}{2}(\phi - \theta) = \frac{1-c}{1+c} \cot \frac{\alpha}{2},$$

$$\tan \tfrac{1}{2}(\phi' - \theta') = \frac{1-c'}{1+c'} \tan \frac{\alpha}{2},$$

$$(\phi + \theta) = \pi - \alpha,$$

$$(\phi' + \theta') = \alpha.$$

These equations determine $\theta$ and $\theta'$, and hence the required angle $\gamma = \theta + \theta'$; also $OR : ON = \sin\phi : \sin\phi'$.

In Table XXII. the angle $\gamma$ and the ratio $ON : OR : = r$ have been calculated for $n = 2$. Table XXIII. gives the corresponding quantities if the inducing solid harmonic is of degree 4.

TABLE XXII.—Comparison between resultant vertical force as regards magnitude and phase when induced currents are taken into account and vertical force calculated on the assumption that the whole is due to an outside effect. The inducing potential is a solid harmonic of degree 2.

| $\delta$. | Reduction in amplitude. $r$. | Change of phase. $\gamma$. | $\rho$. |
|---|---|---|---|
| 1 | ·9981 | 2° 43′ | $3·70 \times 10^{14}$ |
| 5 | ·9565 | 13 02 | $7·40 \times 10^{13}$ |
| 10 | ·8589 | 23 10 | $3·70 \times 10^{13}$ |
| 20 | ·6705 | 34 03 | $1·85 \times 10^{13}$ |
| 30 | ·5516 | 38 12 | $1·23 \times 10^{13}$ |
| 40 | ·4762 | 40 16 | $9·25 \times 10^{12}$ |
| 50 | ·4261 | 41 10 | $7·40 \times 10^{12}$ |
| 100 | ·3004 | 43 08 | $3·70 \times 10^{12}$ |

TABLE XXIII.—Comparison between resultant vertical force, as regards magnitude and direction when induced currents are taken into account, and vertical force calculated on the assumption that the whole is due to an outside effect. The inducing potential is a solid harmonic of degree 4.

| $\delta$. | Reduction in amplitude. | Change of phase. | $\rho$. |
|---|---|---|---|
| | | ° ′ | |
| 1 | ·9997 | 1 06 | $3{\cdot}70 \times 10^{14}$ |
| 10 | ·9723 | 10 06 | $3{\cdot}70 \times 10^{13}$ |
| 100 | ·4982 | 38 49 | $3{\cdot}70 \times 10^{12}$ |

The observed amplitude of the vertical force at Lisbon is about one half of its calculated value. If the conductivity of the Earth was such as to produce this reduction in amplitude, it is seen from Tables XXII. and XXIII. that the phase would be altered about 40°, while in reality there is a remarkable agreement in phase. If the conductivity is so small as to leave the resultant phase practically unaltered, as observation tends to show, the amplitude also should not be sensibly altered. There is, therefore, no uniform conductivity of the Earth which can make the observations agree with the calculation. Such an agreement, however, can be easily brought about, as Professor LAMB has suggested to me, if the conductivity of the inside of the Earth is larger than the conductivity of the upper layers. It is extremely probable that this is really the case. The bulk of the outside layer of the Earth, except in so far as it is water, is made up of material which in its ordinary condition is non-conducting; but we know that some of the silicates begin to conduct at temperatures above 200° C., and, generally speaking, insulators lose their insulating powers at high temperatures. Without regard even to the quantities of metallic matter that may be stored inside the Earth, there is nothing improbable in the supposition that its conductivity increases towards the inside. If the bulk of the observed induced effect is due to currents in a fairly conducting inner sphere, the calculated phase would be that due to good conducting matter, and would not differ from the observed value, while the reduction in amplitude might yet be sufficient to account for the observed facts. In order to give a better idea of the kind of conductivity which is required to produce a certain change of phase, it may be stated that for the purest distilled water obtained by KOHLRAUSCH $\rho$ would be about $1{\cdot}4 \times 10^{15}$. Such water is, as is well known, a very bad conductor, and, according to our Tables, if the whole Earth was made up of matter which conducts as badly, there would be no currents in the Earth induced by the diurnal variation of sufficient intensity to affect our magnetic needle sensibly. Ordinary rain water, however, has a specific resistance of about $6 \times 10^{13}$. A conducting sphere of the same resistance would already produce a retardation in phase of about an hour for the diurnal variation if the solid harmonic is of degree 2. For salt water the resistance may get as low as $4 \times 10^9$. A whole sphere made up of such water would very considerably reduce the amplitude of the observed vertical force, and alter the resultant phase by 45° nearly. The average conductivity of the Earth, as seen from these examples, must be small, although it may be considerable over limited areas. Such limited areas would principally affect the harmonic terms of higher degrees, and we should not consequently expect for them such a good agreement between theory and observation. Table XXII. shows that as the resistance of the sphere diminishes the retardation of the resultant phase seems to approach a constant value of 45°. This can be proved to be quite generally the case. It follows directly from a formula given by Professor LAMB in the Appendix, and can also be seen as follows:—When the conductivity is good the angle between $OH_2$ and $OH_1$ in fig. 11 will steadily diminish, and ultimately vanish. OK will ultimately coincide with $OH_1$, and OR with $OV_1$; but the angle between $OV_2$ and $OV_1$ will increase towards 180°, and the sides will tend towards equality. Two very nearly equal and very nearly opposite forces may have a resultant which is inclined by a finite angle to the forces. To find the angle in the limit between $OV_1$ and $OV_2$ we deduce, in the first place, an expression for the ratio of the vertical force due to the outside effect to the vertical force due to induced currents for good conductivities.

This ratio depends, as shown by Professor LAMB, on the function

$$\chi_n(\kappa R) \quad \text{where} \quad \kappa^2 = \frac{4\pi p i}{\rho},$$

and

$$\chi_n(\zeta) = 3\cdot5 \ldots (2n+1)\left(\frac{d}{\zeta\, d\zeta}\right)^n \frac{\sinh\zeta}{\zeta}.$$

If $n$ is odd,

$$\left(\frac{d}{\zeta\, d\zeta}\right)^n \sinh\zeta = \frac{\cosh\zeta}{\zeta^{n+1}} - \frac{n.n+1}{2}\frac{\sinh\zeta}{\zeta^{n+2}} + \frac{n-1.n.n+1.n+2}{2.4}\frac{\cosh\zeta}{\zeta^{n+3}} - \frac{(n-2)\ldots(n+3)}{2.4.6}\frac{\sinh\zeta}{\zeta^{n+4}}.$$

If $n$ is even, we must interchange $\sinh\zeta$ and $\cosh\zeta$.

If $\zeta$ is larger, so that $e^{-\zeta}$ can be neglected, compared to $e^{\zeta}$, $\cosh\zeta = \sinh\zeta = e^{\zeta}$, and

$$\frac{\kappa^2 R^2}{2n+1.(2n+3)}\frac{\chi_{n+1}(\kappa R)}{\chi_{(n-1)}(\kappa R)}$$

which is the ratio of the vertical forces due to the induced and inducing potential will become

$$\frac{1 - \frac{n+1.n+2}{2}\zeta^{-1} + \frac{n.n+1.n+2.n+3}{2.4}\zeta^{-2} - \ldots}{1 - \frac{n-1.n}{2}\zeta^{-1} + \frac{n-2.n-1.n.n+1}{2.4}\zeta^{-2} - \ldots},$$

where

$$\zeta^2 = \frac{4\pi p}{\rho} R^2 i = i\delta,$$

$$\zeta^{-2} = -\frac{i}{\delta}, \text{ and } \zeta^{-1} = \pm \frac{\delta^{-\frac{1}{2}}}{\sqrt{2}}(1 - i) = \delta^{-\frac{1}{2}}\left(\cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\right).$$

Generally in the above expansion we may put therefore

$$\zeta^{-q} = \delta^{-q/2}\left(\cos\frac{\pi q}{4} - i\sin\frac{\pi q}{4}\right);$$

so that the ratio of the vertical forces will be

$$\frac{X + iY}{X' + iY'},$$

where

$$X = 1 - \frac{n+1.n+2}{2}\cos\frac{\pi}{4}\quad\delta^{-\frac{1}{2}} + \frac{n.n+1.n+2.n+3}{2.4}\cos\frac{2\pi}{4}\quad\delta^{-1} - \frac{(n-1)\ldots(n+4)}{2.4.6}\cos\frac{3\pi}{4}\quad\delta^{-\frac{3}{2}},$$

$$Y = \frac{n+1.n+2}{2}\sin\frac{\pi}{4}\quad\delta^{-\frac{1}{2}} - \frac{n.n+1.n+2.n+3}{2.4}\sin\frac{2\pi}{4}\quad\delta^{-1} + \frac{(n-1)\ldots(n+4)}{2.4.6}\sin\frac{3\pi}{4}\quad\delta^{-\frac{3}{2}},$$

$$X' = 1 - \frac{n-1.n}{2}\cos\frac{\pi}{4}\quad\delta^{-\frac{1}{2}} + \frac{n-2.n-1.n.n+1}{2.4}\cos\frac{2\pi}{4}\quad\delta^{-1} - \ldots,$$

$$Y' = \frac{n-1.n}{2}\sin\frac{\pi}{4}\quad\delta^{-\frac{1}{2}} - \frac{n-2.n-1.n.n+1}{2.4}\sin\frac{2\pi}{4}\quad\delta^{-1} + \ldots$$

The ratio of the amplitudes, if $\delta$ is large, becomes

$$\frac{\sqrt{(X^2 + Y^2)}}{\sqrt{(X'^2 + Y'^2)}} = 1 - (2n+1)(2\delta)^{-\frac{1}{2}},$$

and the angle between the two vertical forces

$$\tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y'}{X'} = (2n+1)(2\delta)^{-\frac{1}{2}}.$$

The resultant of these two forces, resolved in a direction parallel to either of them, is therefore equal to the component which is at right angles, and the resultant will consequently be at an inclination of 45°.



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The result will not hold, of course, if conduction in the Earth's crust takes place chiefly at some distance below the surface, as in that case the vertical force due to the induced currents will not tend to become equal to the vertical force due to the primary variation. For a bad conductivity we shall always have a resultant vertical force sensibly equal to the primary force. If the conductivity increases, the resultant will have a different phase from the primary variation, tending towards a difference of 45°, if the conductivity is uniform. If the conductivity is not uniform, a maximum difference of phase will be reached, which, if the conductivity is still further increased, diminishes indefinitely.

## VI. *The Magnetic Potential on the Surface of the Earth.*

As it seemed interesting to trace the equipotential lines on the surface of the Earth as far as they depend on the diurnal variation, I have calculated the potential from the equations [A] and [B].

It is necessary, for this purpose, to compute the tesseral harmonics for definite points on the Earth's surface. It would seem most natural to choose these points, so that they lie on equidistant circles of latitudes, but as tables* exist for the zonal harmonics in terms of the *cosine* of the colatitude, I have selected values of these cosines so that the corresponding angles should differ as nearly as possible by 10°.

The values of $u$ and $\mu = \cos u$, for which the potential is computed on the Northern hemisphere, are given in Table XXIV., $u$ being the colatitude.

TABLE XXIV.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\mu = \cos u =$ | ·98 | ·94 | ·87 | ·77 | ·64 | ·50 | ·34 | ·17 | ·00 |
| $u =$ | 11° 29′ | 19° 57′ | 29° 32′ | 39° 39′ | 50° 12′ | 60° 00′ | 70° 07′ | 80° 13′ | 90° 00′ |

Symmetrical circles of latitude were taken on the Southern hemisphere.

Taking from Mr. GLAISHER's Table the values of $P_i$ corresponding to each of the above values of $\mu$, we obtain the differential coefficients of zonal harmonics by a successive application of the formula

$$\frac{dP_i}{d\mu} - \frac{dP_{i-2}}{d\mu} = (2i - 1)\, P_{i-1}.$$

The first and second differential coefficients thus calculated are given in Tables XXV. and XXVI.

* 'Report of the British Association' (Sheffield, 1879).

TABLE XXV.—Values of $\frac{\partial P_i}{\partial \mu}$ where $P_i$ is the zonal harmonic of degree $i$ and $\mu$ the cosine of the colatitude.

| $\mu =$ | 1·00. | 0·98. | 0·94. | 0·87. | 0·77. | 0·64. | 0·50. | 0·34. | 0·17. | 0·00. |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{dP_1}{d\mu}$ | + 1·000 | + 1·000 | + 1·000 | + 1·000 | + 1·000 | + 1·000 | + 1·000 | + 1·000 | + 1·000 | + 1·000 |
| $\frac{dP_2}{d\mu}$ | + 3·000 | + 2·940 | + 2·820 | + 2·610 | + 2 310 | + 1·920 | + 1·500 | + 1·020 | + 0·510 | 0·000 |
| $\frac{dP_3}{d\mu}$ | + 6·000 | + 5·705 | + 5·125 | + 4·175 | + 2·945 | + 1·570 | + 0·375 | + 0·635 | − 1·285 | − 1·500 |
| $\frac{dP_4}{d\mu}$ | + 10·000 | + 9·121 | + 7·482 | + 4·997 | + 2·212 | − 0·215 | − 1·559 | − 1·864 | − 1·191 | 0·000 |
| $\frac{dP_5}{d\mu}$ | + 15·000 | + 12·986 | + 9·418 | + 4·562 | + 0·155 | − 2·273 | − 2·226 | − 0·635 | + 1·145 | + 1·875 |
| $\frac{dP_6}{d\mu}$ | + 21·000 | + 17·041 | + 10·496 | + 2·731 | − 2·397 | − 2·943 | − 0·569 | + 1·755 | + 1·856 | 0·000 |
| $\frac{dP_7}{d\mu}$ | + 28·000 | + 21·046 | + 10·393 | − 0·170 | − 4·174 | − 1·428 | + 1·973 | + 2·030 | − 0·662 | − 2·181 |

TABLE XXVI.—Values of $\frac{\partial^2 P_i}{\partial \mu^2}$ where $P_i$ is the zonal harmonic of degree $i$ and $\mu$ the colatitude.

| $\mu =$ | 1·00. | 0·98 | 0·94. | 0·87. | 0·77. | 0·64. | 0·50. | 9·34. | 0·17. | 0·00. |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{d_2 P_2}{d\mu_2}$ | + 3·000 | + 3·000 | + 3·000 | + 3·000 | + 3·000 | + 3·000 | + 3·000 | + 3·000 | + 3·000 | + 3·000 |
| $\frac{d_2 P_3}{d\mu_2}$ | + 15·000 | + 14·700 | + 14·100 | + 13·050 | + 11·550 | + 9·600 | + 7·500 | + 5·100 | + 2·550 | 0·000 |
| $\frac{d_2 P_4}{d\mu_2}$ | + 45·000 | + 42·935 | + 38·875 | + 32·225 | + 23·615 | + 13·990 | + 5·525 | − 1·445 | − 5·995 | − 7·500 |
| $\frac{d_2 P_5}{d\mu_2}$ | + 105·000 | + 96·789 | + 81·438 | + 58·023 | + 31·458 | + 7·665 | − 6·531 | − 11·676 | − 8·269 | 0·000 |
| $\frac{d_2 P_6}{d\mu_2}$ | + 210·000 | + 185·781 | + 142·473 | + 82·407 | + 25·320 | − 11·013 | − 18·861 | − 8·430 | + 6·600 | + 13·125 |
| $\frac{d_2 P_7}{d\mu_2}$ | + 378·000 | + 318·322 | + 217·886 | + 93·526 | + 0·297 | − 30·594 | − 13·928 | + 11·139 | + 15·859 | 0·000 |
| $\frac{d_2 P_8}{d\mu_2}$ | + 630·000 | + 501·471 | + 298·368 | + 79·857 | − 37·290 | − 32·433 | + 10·734 | + 22·020 | − 3·330 | − 19·590 |

Introducing these quantities in the equations for the potential, and taking proper account of the change of sign of $\mu$ in the Southern hemisphere, I have obtained Table XXVII., in which the potentials are given for 24 equidistant meridian circles. In order to reproduce the daily variations, we must imagine the whole system of equipotential lines to revolve round the Earth from East to West; the time for which the potential is given is mean noon for the zero meridian. It will be remembered that the equations for the potential have been derived from the mean summer values in the Northern, and mean winter values in the Southern hemisphere. If we want to get a symmetrical potential in both hemispheres, we must take the average variation for the whole year, or, what comes to the same thing, we may in Table XXVII. write down the mean values for two corresponding circles of latitude, one in each hemisphere. This has been done in Table XXVIII., where the values are only given for the Northern hemisphere. The mean equipotential lines for the year are drawn in fig. 12. If we imagine the variable part of the magnetic force to be produced by a system of surface currents in a conducting sphere concentric with the Earth, and surrounding it, we may, if the potential is known, calculate the distribution of the lines of flow.

If the magnetic surface potential is of the form $\Omega_n$, when $\Omega_n$ is a harmonic of degree $n$, the current function $\phi_n$ is given by

$$4\pi\phi_n = -\frac{2n+1}{n+1}\Omega_n,^*$$

so that the lines of flow are the same as the equipotential lines. This is no longer true when the magnetic potential is made up of a number of terms corresponding to harmonics of different degrees, for the factor $(2n+1)/(n+1)$ will vary for different terms, and the resultant current function will therefore no longer be proportional to the resultant magnetic potential.

In our own case, taking the mean values for the whole year, the series begins with $\Omega_2$, and the factor $(2n+1)/(n+1)$ will vary, therefore, only between 5/3 and 2. We may then, as an approximation, still take the equipotential lines to give us the general form of the lines of flow. We conclude that we may imagine the daily variation of the Earth's magnetic force to be produced by a system of electric currents in a sphere surrounding the Earth, in which the lines of flow are roughly represented in fig. 12, the direction being such that at longitude 60° East the flow is away from the equator.

* MAXWELL, 'Electricity and Magnetism,' vol. 2, p. 281.

TABLE XXVII.—Values of the variable part of the magnetic potential on the cosine of the colatitude) and 24 equidistant meridian circles, reckoned from is that of the mean summer months. The time is Greenwich noon.

| $\mu = \cos u$. | $\lambda = 0$. | 15. | 30. | 45. | 60. | 75. | 90. | 105. | 120. | 135. | 150. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| + ·98 | − 21·7 | − 15·8 | − 9·5 | − 3·3 | + 1·6 | + 5·4 | + 7·9 | + 9·8 | + 11·6 | + 13·5 | + 15·9 |
| + ·94 | − 61·8 | − 46·6 | − 28·6 | −10·6 | + 5·1 | +17·1 | + 25·4 | + 30·7 | + 34·1 | + 37·4 | + 41·0 |
| + ·87 | −135·2 | −106·9 | − 69·5 | −28·8 | + 9·1 | +40·3 | + 62·3 | + 75·3 | + 81·3 | + 83·2 | + 84·1 |
| + ·77 | −212·4 | −171·6 | −113·3 | −47·0 | +17·0 | +70·3 | +107·6 | +127·7 | +133·4 | +129·6 | +122·5 |
| + ·64 | −247·3 | −197·5 | −126·4 | −45·9 | +30·7 | +93·0 | +134·2 | +153·2 | +153·7 | +143·1 | +129·6 |
| + ·50 | −221·2 | −167·3 | − 96·3 | −21·2 | +46·1 | +96·3 | +125·3 | +133·7 | +127·3 | +114·6 | +103·3 |
| + ·34 | −158·1 | −105·7 | − 45·3 | +11·8 | +56·3 | +83·4 | + 92·5 | + 88·0 | + 76·9 | + 67·0 | + 64·1 |
| + ·17 | − 93·5 | − 49·5 | − 5·1 | +31·2 | +54·2 | +62·4 | + 58·2 | + 47·2 | + 36·0 | + 30·8 | + 34·7 |
| 00 | − 42·4 | − 13·4 | + 13·1 | +32·2 | +41·6 | +41·5 | + 34·8 | + 24·9 | + 17·0 | + 14·4 | + 18·5 |
| − ·17 | + 6·4 | + 17·1 | + 24·1 | +26·8 | +24·7 | +19·6 | + 13·0 | + 6·8 | + 2·9 | + 1·4 | + 2·5 |
| − ·34 | + 62·3 | + 56·2 | + 43·3 | +26·3 | + 8·4 | − 7·1 | − 18·5 | − 24·5 | − 26·0 | − 24·5 | − 22·3 |
| − ·50 | +111·2 | + 94·3 | + 65·6 | +30·4 | − 5·2 | −35·6 | − 56·7 | − 67·1 | − 67·2 | − 61·0 | − 52·2 |
| − ·64 | +129·2 | +107·8 | + 72·3 | +28·1 | −17·3 | −56·7 | − 84·9 | − 98·7 | − 98·9 | − 88·7 | − 73·1 |
| − ·77 | +106·8 | + 85·4 | + 52·2 | +11·7 | −29·7 | −65·9 | − 92·1 | −105·1 | −104·9 | − 93·9 | − 76·2 |
| − ·87 | + 63·0 | + 44·2 | + 19·1 | − 9·2 | −37·2 | −61·1 | − 78·0 | − 85·9 | − 84·8 | − 76·5 | − 61·7 |
| − ·94 | + 26·3 | + 12·1 | − 3·8 | −20·1 | −35·1 | −47·1 | − 55·0 | − 58·1 | − 56·3 | − 50·1 | − 40·4 |
| − ·98 | + 8·6 | − 0·2 | − 9·2 | −17·5 | −24·6 | −29·9 | − 33·1 | − 33·7 | − 32·0 | − 28·1 | − 22·2 |

TABLE XXVIII.—Values of the variable part of the magnetic potential on the cosine of the colatitude) and 24 equidistant meridian circles, reckoned from is that corresponding to the mean values for the year. The time is Green-

| $\mu = \cos u$. | $\lambda = 0$. | 15. | 30. | 45. | 60. | 75. | 90. | 105. | 120. | 135. | 150. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ·98 | − 15·1 | − 7·8 | − 0·1 | + 7·1 | +13·1 | +17·6 | + 20·5 | + 21·7 | + 21·8 | + 20·8 | + 19·0 |
| ·94 | − 44·0 | − 29·3 | − 12·4 | + 4·7 | +20·1 | +32·1 | + 40·2 | + 44·4 | + 45·2 | + 43·7 | + 40·7 |
| ·87 | − 99·1 | − 75·5 | − 44·3 | − 9·8 | +23·1 | +50·7 | + 70·1 | + 80·6 | + 83·0 | + 79·8 | + 72·9 |
| ·77 | −159·6 | −128·5 | − 82·7 | −29·3 | +23·3 | +68·1 | + 99·8 | +116·4 | +119·1 | +111·7 | + 99·3 |
| ·64 | −188·2 | −152·6 | − 99·3 | −37·0 | +24·0 | +74·8 | +109·5 | +125·9 | +126·3 | +115·9 | +101·3 |
| ·50 | −166·2 | −130·8 | − 80·9 | −25·8 | +25·6 | +65·9 | + 91·0 | +100·4 | + 97·2 | + 87·8 | + 77·7 |
| ·34 | −110·2 | − 80·9 | − 44·3 | − 7·2 | +23·9 | +45·2 | + 55·5 | + 56·2 | + 51·4 | + 45·7 | + 43·2 |
| ·17 | − 49·9 | − 33·3 | − 14·6 | + 2·2 | +14·7 | +21·4 | + 22·6 | + 20·2 | + 16·5 | + 14·7 | + 16·1 |
| ·00 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 |

Earth's surface for 17 latitude circles corresponding to different values of $\mu$ (the Greenwich towards the East. The period of the year to which the Table refers

| 165. | 180. | 195. | 210. | 225. | 240. | 255. | 270. | 285. | 300. | 315. | 330. | 345. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + 18·4 | + 20·3 | + 21·0 | + 19·7 | + 16·1 | +10·2 | + 2·2 | − 6·5 | −15·0 | − 21·8 | − 26·3 | − 27·7 | − 26·0 |
| + 45·2 | + 48·8 | + 50·4 | + 48·4 | + 40·8 | +27·5 | + 9·1 | −12·4 | −34·5 | − 53·9 | − 67·6 | − 73·6 | − 71·4 |
| + 85·7 | + 87·6 | + 98·3 | + 85·4 | + 74·2 | +54·1 | +23·0 | −14·7 | −56·7 | − 96·9 | −128·6 | −147·3 | −149·5 |
| +116·6 | +114·0 | +113·4 | +110·9 | +101·0 | +79·0 | +41·9 | − 9·2 | −69·5 | −131·0 | −183·6 | −218·5 | −228·8 |
| +119·7 | +116·3 | +118·2 | +119·8 | +113·9 | +93·5 | +54·4 | − 3·2 | −73·8 | −147·1 | −211·1 | −253·8 | −267·1 |
| + 98·9 | +102·6 | +111·1 | +118·0 | +114·2 | +93·7 | +53·7 | − 6 7 | −77·5 | −148·7 | −207·6 | −243·2 | −247·9 |
| + 70·7 | + 84·5 | + 99·9 | +109·1 | +104·4 | +81·1 | +38·4 | −18·9 | −82·2 | −140·7 | −183·2 | −201·5 | −192·5 |
| + 47·5 | + 64·9 | + 80·9 | + 88·3 | + 81·2 | +57·4 | +18·6 | −29·6 | −78·6 | −119·2 | −143·2 | −146·3 | −128·5 |
| + 27·8 | + 39·4 | + 48·6 | + 50·9 | + 43·4 | +25·2 | − 1·1 | −31·5 | −60·1 | − 81·0 | − 90·0 | − 85·3 | − 68·2 |
| + 5·1 | + 7·4 | + 7·9 | + 5·1 | − 0 8 | − 9·1 | −18·6 | −26·8 | −31·8 | − 32·1 | − 27·4 | − 18·1 | − 6·1 |
| − 21·4 | − 22·7 | − 26·6 | − 31·5 | − 35·5 | −36·2 | −31·7 | −21·1 | − 5·1 | + 14·2 | + 33·7 | + 50·1 | + 60·2 |
| − 44·7 | − 41·0 | − 41·5 | − 44·6 | − 46·6 | −43·8 | −33·2 | −13·5 | +14·2 | + 46·2 | + 77·2 | +101·2 | +113·5 |
| − 57·2 | − 45·0 | − 38·2 | − 35·7 | − 34·5 | −30·1 | −19·3 | + 0·7 | +29·1 | + 62·3 | + 95·1 | +120·5 | +133·2 |
| − 56·6 | − 38·8 | − 25·4 | − 16·8 | − 9·1 | − 1·9 | + 8·5 | +24·1 | +44·9 | + 68·5 | + 91·3 | +107·8 | +114·0 |
| − 44·8 | − 27·6 | − 12·2 | + 0·9 | + 12·0 | +22·0 | +31·9 | +42·6 | +53·9 | + 64·8 | + 73·2 | + 76·8 | + 74·0 |
| − 28·7 | − 15·7 | − 2·7 | + 9·4 | + 20·5 | +29·9 | +38·1 | +44·4 | +48·7 | + 50·7 | + 49·7 | + 45·6 | + 37·7 |
| − 15·0 | − 6·8 | + 1·4 | + 9·6 | + 17·1 | +23·4 | +28·3 | +31·3 | +32·5 | + 31·6 | + 28·5 | + 23·4 | + 16·6 |

Earth's surface for 17 latitude circles corresponding to different values of $\mu$ (the Greenwich towards the East. The period of the year to which the Table refers wich noon.

| 165. | 180. | 195. | 210. | 225. | 240. | 255. | 270. | 285. | 300. | 315. | 330. | 345. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| +16·7 | +13·5 | + 9·8 | + 5·0 | − 0·5 | − 6·6 | −13·0 | −18·9 | −23·7 | − 26·7 | − 27·4 | − 25·5 | − 21·3 |
| +36·9 | +32·2 | +26·5 | +19·5 | +10·1 | − 1·2 | −14·5 | −28·4 | −41·6 | − 52·3 | − 58·6 | − 59·6 | − 54·5 |
| +65·2 | +57·6 | +55·2 | +42·2 | +31·1 | +16·0 | − 4·4 | −28·6 | −55·3 | − 80·8 | −100·9 | −112·0 | −111·7 |
| +86·6 | +76·4 | +69·4 | +63·8 | +55·0 | +40·4 | +16·7 | −16·6 | −57·2 | − 99·7 | −137·4 | −163·1 | −171·4 |
| +88·4 | +80·6 | +78·2 | +77·7 | +74·2 | +61·8 | +36·8 | − 1·9 | −51·4 | −104·7 | −153·1 | −187·1 | −200·1 |
| +71·8 | +71·8 | +76·3 | +81·3 | +80·4 | +68·7 | +43·4 | + 3·4 | −45·8 | − 97·4 | −142·4 | −172·2 | −180·7 |
| +46·0 | +53·6 | +63·2 | +70·3 | +69·9 | +58·6 | +35·0 | + 1·1 | −38·5 | − 77·4 | −108·4 | −125·8 | −126·3 |
| +21·2 | +28·7 | +36·5 | +41·6 | +41·0 | +33·2 | +18·6 | − 1·4 | −23·4 | − 43·5 | − 57·9 | − 64·1 | − 61·2 |
| 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 | 00·0 |

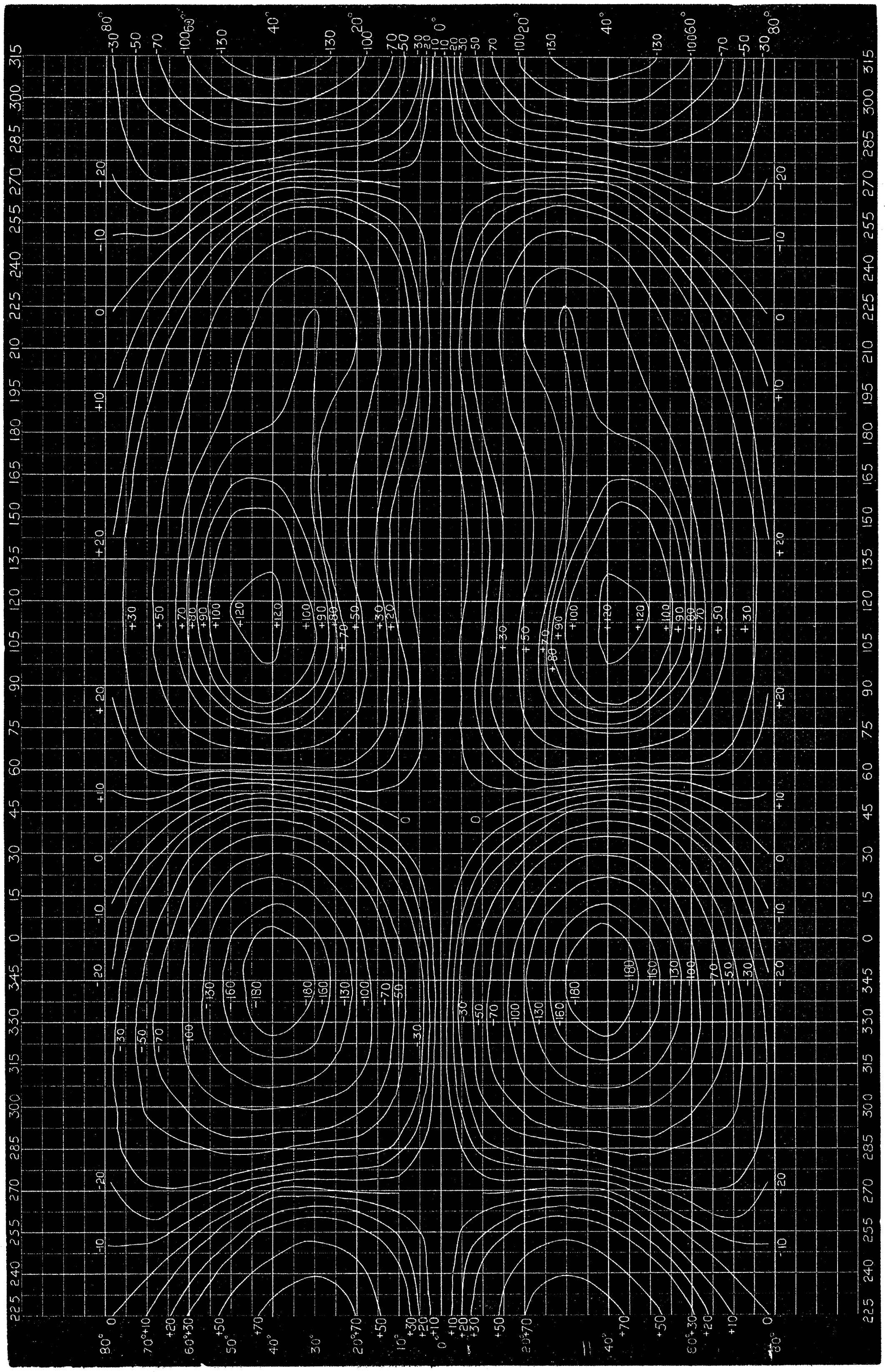

Fig. 12.



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## VII. *Concluding Remarks.*

FARADAY, in the year 1850, discussed the diurnal variation of the magnetic needle. He showed that the changes which took place during daytime could be accounted for by supposing two magnetic poles—namely, a North pole in the Southern hemisphere, and a South pole in the Northern hemisphere—to be carried round with the Sun in our atmosphere. A glance at fig. 13 will show that our result entirely agrees with FARADAY'S. The proof that the principal part of the Earth's magnetism is due to causes outside its surface would have been almost as complete in the year 1850 as it is now, if FARADAY had added the remark that, if all three components of the variation can be completely accounted for by hypothetical changes taking place outside the Earth's surface, they *cannot* be accounted for by changes taking place in the inside.

I cannot agree, however, with FARADAY in the explanation which he gives of the variation. He imagines that the solar radiation, heating up the air, produces a sufficient change in its magnetic permeability to account for the observed deflection of the lines of magnetic force.

The magnetic susceptibility of oxygen at the atmospheric pressure and temperature is about ·5, $10^{-7}$, and for air it is smaller still. This would give the magnetic permeability as 1·0000006. If the air was entirely removed the change of magnetic force would be so small that we could not detect it. I have tried in various ways to find how a partial removal of the atmosphere as a magnetic medium could affect the needle in any appreciable way, but have failed to do so. FARADAY suggests that the oxygen in the higher regions of the atmosphere might, owing to the greater cold, be much more magnetic than what we observe it to be. But, on the other hand, owing the smaller density, the permeability would be diminished; so that I do not think we are at present justified in ascribing any material part of the daily variation to a change of the magnetic permeability of air due to the heating effect of the Sun. The effect of the Moon suggests a tidal action as the cause, and we may inquire whether such a tidal action could produce the observed effects. The late Professor BALFOUR STEWART has suggested that the Earth's magnetic force might induce electric currents in the convection currents which flow in the upper regions of the atmosphere. One difficulty of this hypothesis was removed by an experimental investigation, by means of which I have proved that the air can be thrown into a sensitive state in which small electromotive forces will produce sensible electric currents. To bring the air into that sensitive state it is only necessary to send an electric current through it from some independent source of high potential. It is very likely that the air in the upper regions of our atmosphere is in such a sensitive state, and it is quite possible, therefore, that the induced electric currents suggested by Professor BALFOUR STEWART really exist.

The symmetry of the diurnal variation in both hemispheres shows that, if it is due to the assumed cause, the vertical component of the magnetic force is the important factor, as that component changes sign on crossing the magnetic equator. In order that electric currents should be induced which could account for the observed movement of the magnetic needle, it is only necessary to imagine convection currents in the upper regions from East to West during certain parts of the day, and from West to East at other times. Judging from the analogy of the theory of waves in shallow water, a horizontal motion of considerable velocity might be produced by a tidal action due to solar and lunar attraction. It is true that no periodic change of the barometer has been traced with certainty to a tidal action; but I suppose that a tidal wave must nevertheless exist, and that its horizontal flow might be considerable, while the changes of pressure might escape our attention. As regards the effect of the Sun we have, indeed, a daily period of the barometer which is probably due to thermal effects. It is curious and suggestive that the horizontal motion which must accompany the change in pressure is just such as would account for the daily variation of the magnetic needle. In the tropics the principal minimum of the barometer takes place about 3.40 o'clock in the afternoon, and the principal maximum about 9 o'clock in the morning. According to the theory of waves, there would be a horizontal movement from West to East in the afternoon, and from East to West in the morning. The direction of the induced electric currents would be away from the equator in both hemispheres in the afternoon, and towards the equator in the morning. This is exactly the system of currents we have been led to, starting from the observed magnetic variation. The only difficulty I feel in suggesting that the cause of the diurnal variation of the magnetic needle is the diurnal variation of the barometer lies in the fact that it would oblige us to place the electric currents into the lower regions of the atmosphere, as these only will be much affected by the thermal radiation of the Sun. The phase of the barometric oscillation has been found to be reversed on the top of mountains, and it would be interesting to see whether the magnetic variations show any peculiarities at great heights.*

The region of the atmosphere which other considerations lead me to consider as the most sensitive to electromotive forces is that of the cirrus clouds, and I should be inclined to look to that region for a solution of the question. The lunar action seems, according to the researches of Mr. CHAMBERS, to be a modification of the solar action rather than an independent effect. This might be accounted for if we suppose that the conductivity of the air depends on the position of the Sun, while the electromotive forces depend on the combined positions of the Sun and Moon.

* [Note added October 11, 1889.—Since writing the above I have become acquainted with HANN's recent work on the diurnal oscillation of the barometer ('Wien, Denkschriften,' vol. 55, 1889.) It appears from the regularity of the semidiurnal period in different altitudes and latitudes that its cause must lie in atmospheric movements in higher regions of the atmosphere. The reversal of phase mentioned in the text is due to local effects and has nothing to do with the regular oscillation. It seems to be exceedingly probable in the light of these researches that the daily variation of the magnetic needle is connected with the daily oscillation of the barometer in the way described in the text.]

It will be interesting to follow out in future researches the field which this investigation has opened, especially in order to trace the effect of the Sunspot variation; but for this purpose it is absolutely necessary that different observatories should follow a more uniform plan in reducing their observations. It has been found by experience that if the hourly readings of the magnetic needle are collected together, and their mean taken, that mean is different according as the disturbed days are taken into account or rejected. In other words, the disturbances are not irregularly distributed, but have a daily period which is mixed up with the regular daily variation. If we want to separate the investigation concerning the regular variation and the disturbance variation, we must adopt some plan of obtaining the one without the other. I need not here describe SABINE's well-known method of doing this. Grave objections have been urged against it, but it is still adopted in many observatories. A discussion of the various methods of reduction which have been proposed will be found in recent Reports of the British Association, and amongst them that adopted by Mr. WILD at St. Petersburg seems to me to be the only one which can be justified on strict scientific principles. It consists in selecting the curves for the quiet days, of which there are always a sufficient number in each month, and not to take account, as SABINE's method does, of any reading at all during the disturbed days. We get in this way something perfectly definite, namely, the mean variation of the magnetic needle during certain specified days. It seems to me that if the heads of different observatories could adopt some system of intercommunication, by which they could select those days which are most quiet all over the world, and if the elements are reduced for those days solely at the different stations, we should obtain a series of values for different points of the Earth which are strictly comparable with each other. The labour of reduction, as far as I can judge, would thereby be seriously diminished. The method hitherto adopted at Greenwich is very similar to that of WILD, and will not, probably, lead to results which are sensibly different.

The reduction of the observations by spherical harmonic analysis would be a very simple matter according to the method which I have followed, if the results of different stations were published in a manner which would lend itself easily to the work. The method of publication adopted at Greenwich is very convenient, and might serve as a model to other observatories. Much labour is, however, involved in reducing variations in horizontal force and declination to variations in force towards the geographical West and North respectively. If all observatories could publish the coefficients of the harmonic series of the elements as at Greenwich, but reduced to the geographical instead of the magnetic co-ordinates, the progress of magnetic science would be much assisted, as every scientific investigation must take the geographical components for its starting point.

I have tried to form an idea as to the degree of accuracy reached in the determination of such quantities as the daily variation of declination; the result is not altogether satisfactory. Mr. WHIPPLE, in the 'British Association Report' (Birmingham, 1886, page 71), says:—

"Contrasting the Kew results with those of Greenwich, we may fairly consider the difference to be due in some measure to instrumental causes, the construction of the magnetographs being dissimilar at the two observatories. The slight difference in position of the two observatories may likewise have some influence."

The difference amounts to about 15 per cent., and it seems as if the question whether such a difference can be due to instrumental causes deserves a careful examination. Mr. CHAMBERS, at Bombay, has found similarly that the results of the magnetographs differ from those obtained by the old magnetometers; and he seems to ascribe the difference to an "influence of height above or below the ground level." The height of the magnetometer was 6 feet above ground, and that of the magnetograph $7\frac{1}{2}$ feet below; the former gives ranges greater by 7 per cent. for the declination variation, and the difference is greater still for the horizontal force component. That there should be a real difference of that magnitude in the two positions seems excessively unlikely, and we must conclude that at present the results given by magnetographs are doubtful to the extent of about 10 per cent.

In conclusion, we may sum up the principal results obtained in this paper as follows:—

1. The principal part of the diurnal variation is due to causes outside the Earth's surface, and probably to electric currents in our atmosphere.

2. Currents are induced in the Earth by the diurnal variation which produce a sensible effect chiefly in reducing the amplitude of the vertical component and increasing the amplitude of the horizontal components.

3. As regards the currents induced by the diurnal variation, the Earth does not behave as a uniformly-conducting sphere, but the upper layers must conduct less than the inner layers.

4. The horizontal movements in the atmosphere which must accompany a tidal action of the Sun or Moon or any periodic variation of the barometer such as is actually observed would produce electric currents in the atmosphere having magnetic effects similar in character to the observed daily variation.

5. If the variation is actually produced by the suggested cause, the atmosphere must be in that sensitive state in which, according to the author's experiments, there is no lower limit to the electromotive force producing a current.

In conclusion, the author begs to return his thanks to Mr. WILLIAM ELLIS for help given in some of the calculations, and also to his assistant, Mr. ARTHUR STANTON, for much labour bestowed on making and checking numerical calculations.

## APPENDIX.

### *On the Currents Induced in a Spherical Conductor by Variation of an External Magnetic Potential.*

*By* HORACE LAMB, *M.A., F.R.S.*

THE general formulæ for the currents induced in a sphere of uniform conductivity by any electric or magnetic disturbances outside it have been given in the 'Phil. Trans.,' 1883, pp. 526 *et seq.* I here reproduce (with some further developments) so much of the investigation as is required for the discussion in Part V. of the foregoing paper.

Suppose that, the origin being taken at the centre of the Earth, we have an external disturbing force, whose magnetic potential near the Earth's surface is

$$\Omega_n e^{i(pt+\epsilon)},$$

when $\Omega_n$ is a solid harmonic of positive degree $n$, and $i = \sqrt{-1}$.

The corresponding values of the components of electric momentum are

$$F = \frac{1}{n+1}\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\Omega_n,$$

$$G = \frac{1}{n+1}\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\Omega_n,$$

$$H = \frac{1}{n+1}\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\Omega_n,$$

the time-factor being omitted here and elsewhere for shortness. For these values make

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0,$$

and also give, for the components of magnetic force, the values

$$\alpha = \frac{dH}{dy} - \frac{dG}{dz} = \frac{1}{n+1}\left\{x\nabla^2\Omega_n - \left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + 2\right)\frac{d\Omega_n}{dx}\right\}$$

$$= -\frac{d\Omega_n}{dx},$$

and, similarly,

$$\beta = -\frac{d\Omega_n}{dy},$$

$$\gamma = -\frac{d\Omega_n}{dz}.$$

If $u, v, w$ be the components of electric current, the equations of induced currents are

$$\rho u = -\dot{F}, \qquad \rho v = -\dot{G}, \qquad \rho w = -\dot{H},$$

where $\rho$ denotes the specific resistance. Eliminating $u, v, w$ by means of the relations

$$\nabla^2 F = -4\pi u, \qquad \nabla^2 G = -4\pi v, \qquad \nabla^2 H = -4\pi w,$$

we find that at all points within the sphere F, G, H must satisfy the equations

$$(\nabla^2 - k^2) F = 0, \qquad (\nabla^2 - k^2) G = 0, \qquad (\nabla^2 - k^2) H = 0,$$

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0,$$

where

$$k^2 = \frac{4\pi p}{\rho} \cdot i.$$

The appropriate solution of these is

$$\left.\begin{aligned} F &= \chi_n (kr) . \left(y \frac{d}{dz} - z \frac{d}{dy}\right) \omega_n, \\ G &= \chi_n (kr) . \left(z \frac{d}{dx} - x \frac{d}{dz}\right) \omega_n, \\ H &= \chi_n (kr) . \left(x \frac{d}{dy} - y \frac{d}{dx}\right) \omega_n, \end{aligned}\right\}$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, $\omega_n$ denotes a solid harmonic of degree $n$, and

$$\chi_n (\zeta) = 1 + \frac{\zeta^2}{2.2n + 3} + \frac{\zeta^4}{2.4.2n + 3.2n + 5} + \ldots$$

$$= 3.5 \ldots 2n + 1 . \left(\frac{d}{\zeta d\zeta}\right)^n \frac{\sinh \zeta}{\zeta}.$$

The total magnetic potential outside the sphere will be

$$\Omega_n + \Omega_{-n-1},$$

where $\Omega_{-n-1}$ is the part due to the induced currents. The values of F, G, H at external points will then be

$$F = \left(y \frac{d}{dz} - z \frac{d}{dy}\right) \left(\frac{1}{n + 1} \Omega - \frac{1}{n} \Omega_{-n-1}\right),$$

$$G = \left(z \frac{d}{dx} - x \frac{d}{dz}\right) \left(\frac{1}{n + 1} \Omega_n - \frac{1}{n} \Omega_{-n-1}\right),$$

$$H = \left(x \frac{d}{dy} - y \frac{d}{dx}\right) \left(\frac{1}{n + 1} \Omega_n - \frac{1}{n} \Omega_{-n-1}\right).$$

It remains to introduce the conditions to be satisfied at the surface of the sphere ($r = \mathrm{R}$). The continuity of electromotive force, *i.e.*, of $\dot{\mathrm{F}}$, $\dot{\mathrm{G}}$, $\dot{\mathrm{H}}$, requires

$$\chi_n(k\mathrm{R}) \cdot \omega_n = \frac{1}{n+1}\,\Omega_n - \frac{1}{n}\,\Omega_{-n-1}. \quad [r = \mathrm{R}]$$

The continuity of the magnetic force involves the continuity of the space-derivatives of F, G, H, and, therefore, of $d\mathrm{F}/dr$, $d\mathrm{G}/dr$, $d\mathrm{H}/dr$. Hence

$$\{k\mathrm{R}\chi'_n(k\mathrm{R}) + n\chi_n(k\mathrm{R})\}\,\omega_n = \frac{n}{n+1}\,\Omega_n + \frac{n+1}{n}\,\Omega_{-n-1}. \quad [r = \mathrm{R}]$$

We thence find

$$\left.\begin{aligned} k\mathrm{R}\,.\,\chi'_n(k\mathrm{R})\,.\,\omega_n &= \frac{2n+1}{n}\,\Omega_{-n-1}, \\ \{k\mathrm{R}\chi'_n(k\mathrm{R}) + (2n+1)\,\chi_n(k\mathrm{R})\}\,\omega_n &= \frac{2n+1}{n+1}\,\Omega_n, \end{aligned}\right\} \quad [r = \mathrm{R}]$$

which are equivalent to

$$\left.\begin{aligned} \frac{k^2\mathrm{R}^2}{2n+1\,.\,2n+3}\,\chi_{n+1}(k\mathrm{R})\,.\,\omega_n &= \frac{1}{n}\,\Omega_{-n-1}, \\ \chi_{n-1}(k\mathrm{R})\,.\,\omega_n &= \frac{1}{n+1}\,\Omega_n. \end{aligned}\right\} \quad [r = \mathrm{R}]$$

Hence

$$\frac{\Omega_{-n-1}}{\Omega_n} = \frac{n}{n+1} \cdot \frac{k^2\mathrm{R}^2}{2n+1\,.\,2n+3} \cdot \frac{\chi_{n+1}(k\mathrm{R})}{\chi_{n-1}(k\mathrm{R})}.$$

This gives the ratio of surface potentials, and, therefore, of horizontal forces, due to internal and external influences respectively. Since this ratio is "complex," there will be a difference of phase, as well as of amplitude. The corresponding ratio of vertical forces is

$$\left[\frac{-\dfrac{d\Omega_{-n-1}}{dr}}{-\dfrac{d\Omega_n}{dr}}\right]_{r=\mathrm{R}} = -\frac{n+1}{n} \cdot \frac{\Omega_{-n-1}}{\Omega_n} = -\frac{k^2\mathrm{R}^2}{2n+1\,.\,2n+3} \cdot \frac{\chi_{n+1}(k\mathrm{R})}{\chi_{n-1}(k\mathrm{R})}.$$

To interpret these results it is necessary to calculate the function

$$\frac{\zeta^2}{2n+1\,.\,2n+3} \cdot \frac{\chi_{n+1}(\zeta)}{\chi_{n-1}(\zeta)},$$

where

$$\zeta^2 = k^2\mathrm{R}^2 = \frac{4\pi p\mathrm{R}^2}{\rho} \cdot i = i\delta, \text{ say.}$$

For moderate values of δ we may use the form

$$\frac{i\delta}{2n+1.2n+3}\cdot\frac{A_{n+1}+iB_{n+1}}{A_{n-1}+iB_{n-1}},$$

where

$$A_n = 1 - \frac{\delta^2}{2.4.2n+3.2n+5} + \frac{\delta^4}{2.4.6.8.2n+3.2n+5.2n+7.2n+9} - \ldots,$$

$$B_n = \frac{\delta}{2.2n+3} - \frac{\delta^3}{2.4.6.2n+3.2n+5.2n+7} = \ldots.$$

The following Table gives the values of $A_1$, $B_1$, $A_3$, $B_3$, $A_5$, $B_5$, for various values of δ. It may possibly be of service in other investigations.

TABLE XXIX.

| δ. | $A_1$. | $B_1$. | $A_3$. | $B_3$. | $A_5$. | $B_5$. |
|---|---|---|---|---|---|---|
| 1 | + 0·996429 | + 0·099934 | + 0·998738 | + 0·055539 | + 0·999359 | + ·038456 |
| 2 | + 0·985726 | + 0·199471 | + 0·994952 | + 0·110982 | | |
| 3 | + 0·967918 | + 0·298216 | + 0·988647 | + 0·116230 | | |
| 4 | + 0·943050 | + 0·395773 | + 0·979832 | + 0·221187 | | |
| 5 | + 0·911184 | + 0·491751 | + 0·968518 | + 0·275757 | | |
| 6 | + 0·872401 | + 0·585759 | + 0·954720 | + 0·329843 | | |
| 7 | + 0·826800 | + 0·677412 | + 0·938455 | + 0·383350 | | |
| 8 | + 0·774498 | + 0·766327 | + 0·919714 | + 0·436182 | | |
| 9 | + 0·715628 | + 0·852126 | + 0·898610 | + 0·488246 | | |
| 10 | + 0·650341 | + 0·934439 | + 0·875083 | + 0·539448 | + 3·936309 | + ·3784 |
| 20 | − 0·310366 | + 1·489206 | + 0·516311 | + 0·984135 | | |
| 30 | − 1·628644 | + 1·351846 | − 0·029614 | + 1·248628 | | |
| 40 | − 2·912855 | + 0·337304 | − 0·688914 | + 1·265555 | | |
| 50 | − 3·716054 | − 1·563371 | − 1·366602 | + 0·993199 | | |
| 100 | + 12·514840 | − 10·691120 | − 1·308236 | − 3·811231 | − 1·9434 | − ·6646 |

If we write

$$A_n + iB_n = S_n e^{i\phi_n},$$

the above fraction becomes

$$\frac{\delta}{2n+1.2n+3}\cdot\frac{S_{n+1}}{S_{n-1}}\cdot e^{i\left(\phi_{n+1}-\phi_{n-1}+\frac{1}{2}\pi\right)}.$$

If we prefix the *minus* sign this gives the ratio ($k'$) of the vertical forces. To get the ratio ($k$) of the horizontal forces we must multiply by $n/(n+1)$.

For large values of δ we may make use of the second form of $\chi_n$. Thus

$$\chi_1 = 3\left\{\frac{\cosh\zeta}{\zeta^2} - \frac{\sinh\zeta}{\zeta^3}\right\},$$

$$\chi_3 = 3.5.7\left\{\left(\frac{1}{\zeta^4}+\frac{15}{\zeta^6}\right)\cosh\zeta - \left(\frac{6}{\zeta^5}+\frac{15}{\zeta^7}\right)\sinh\zeta\right\},$$

whence

$$\frac{\zeta^2}{5.7}\frac{\chi_3(\zeta)}{\chi_1(\zeta)} = \frac{(1 + 15\zeta^{-2})\cosh\zeta - (6\zeta^{-1} + 15\zeta^{-3})\sinh\zeta}{\cosh\zeta - \zeta^{-1}\sinh\zeta}.$$

Here

$$\zeta = (i\delta)^{\frac{1}{2}} = (1 + i)\,\delta^{\frac{1}{2}}/\sqrt{2} = (1 + i)\,\beta, \text{ say.}$$

If $\beta$ is moderately large we may put $\cosh\zeta/\sinh\zeta = 1$ approximately. The error thus committed is of the order $e^{-2\beta}$. Since the value of $e^{-14}$ has six cyphers after the decimal point, this approximation is amply sufficient for $\beta > 7$, or say for $\delta > 100$. The above fraction is then

$$= \frac{1 - 6\zeta^{-1} + 15\zeta^{-2} - 15\zeta^{-3}}{1 - \zeta^{-1}} = \frac{(1 - 3\beta^{-1} + \frac{15}{4}\beta^{-3}) + i(3\beta^{-1} - \frac{15}{2}\beta^{-2} + \frac{15}{4}\beta^{-3})}{(1 - \frac{1}{2}\beta^{-1}) + i\,.\,\frac{1}{2}\beta^{-1}}.$$

It is by these methods that Tables XX. and XXI. above were calculated. The values of $\rho$, the specific conductivity, given in the fifth columns, were obtained from the formula

$$\rho = 4\pi p \mathrm{R}^2/\delta$$

by putting

$$2\pi\mathrm{R} = 4{\cdot}10^9 \text{ cm.}, \quad 2\pi/p = 86{,}400/m \text{ secs.},$$

where $m$ denotes the number of complete periods in a day, and is therefore $= 1$ for the diurnal and $= 2$ for the semidiurnal variations.

As the resistance diminishes, the difference of phase tends to zero, and the ratio of normal forces to the value $-1$; *i.e.*, the total normal force at the surface tends to zero, in accordance with the theory of electromagnetic screens.

The ratio of the total vertical to the total horizontal force in any assigned direction is

$$\frac{-\dfrac{d}{dr}(\Omega_n + \Omega_{-n-1})}{-\dfrac{d}{\mathrm{R}d\eta}(\Omega_n + \Omega_{-n-1})}$$

where $\mathrm{R}d\eta$ denotes a linear element drawn in the proper direction on the Earth's surface. By means of the preceding results this can be put in the form

$$\frac{n(n+1)\,\chi_n(k\zeta)}{k\mathrm{R}\,\chi'_n(k\mathrm{R}) + (n+1)\,\chi_n(k\mathrm{R})}\cdot\Omega_n\Big/\frac{d\Omega_n}{d\eta}.$$

The coefficient may be calculated independently, by a proper adaptation of the previous methods, or we may deduce its value from the results already obtained, in the manner explained by Professor SCHUSTER. The function actually computed by Professor SCHUSTER in Tables XXII. and XXIII. is the ratio

$$\frac{-\frac{d}{dr}(\Omega_n + \Omega_{-n-1})}{-\frac{d\Omega_n}{dr}},$$

which

$$= \frac{(n+1)\chi_n(k\text{R})}{k\text{R}\chi_n'(k\text{R}) + (n+1)\chi_n(k\text{R})}.$$

For large values of $\delta$, *i.e.* for sufficiently small values of $\rho$, we may put $\cosh\zeta = \sinh\zeta$, whence, keeping only the most important term in $\chi_n(\zeta)$, the fraction written becomes

$$= (n+1)/\zeta = \frac{n+1}{\sqrt{\delta}} \cdot e^{-i\pi/4}.$$

The difference of phase therefore tends to the limit 45°, as remarked by Professor SCHUSTER. For $\delta = 100$, this formula gives for the reduction of amplitude the value ·3 and ·5 in the cases $n = 2$ and $n = 4$ respectively.



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